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A local property of basic Rickard equivalences

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ABSTRACT

Let p be a prime and k be an algebraically closed field of characteristic p. Let G and G' be two finite groups, b and b' be blocks of G and G' and (P,e) and (P',e') be maximal Brauer (b,G)- and (b',G')-pairs. If kGb and $kG'b'$ are basic Rickard equivalent, then there is a group isomorphism $\lambda\colon P \cong P'$ such that it induces an equivalence between the Brauer categories of kGb and $kG'b'$ and that, for any Brauer (b,G)-pair (Q,f) contained in (P,e) and any subgroup K of the image of $N_G(Q,f)$ in $\mathrm{Aut}(Q)$, the block algebras $kN_G^K(Q)f$ and $kN_{G'}^{K'}(Q')f'$ are basic Rickard equivalent, where $Q' = \lambda(Q)$, f' is the block of $kC_{G'}(Q')$ such that $(Q',f') \leqslant (P',e')$, and K' denotes the image of K in $\mathrm{Aut}(Q')$.

1. Introduction

1.1. Let p be a prime and k an algebraically closed field of characteristic p. If two block algebras kGb and $kG'b'$ of two finite groups G and G' are *basic Morita equivalent* — namely, they are Morita equivalent *via* an indecomposable $k(G \times G')$-module having a *vertex* $\ddot{P} \subset G \times G'$ and a *source* $\ddot{N}$ such that $\ddot{P}$ stabilizes a k-basis of $\mathrm{End}_k(\ddot{N})$ [6, Section 7] — then there exists a group isomorphism $P \cong P'$ between their defect groups P and P', inducing an equivalence between their *local categories* [6, 7.6.6]. In this case, we have proved in [7] that for any local pointed group Q_δ on kGb, denoting by $Q'_{\delta'}$ the local pointed group on $kG'b'$ corresponding to Q_δ throughout this equivalence, the respective blocks b_δ and $b_{\delta'}$ of the normalizers $N_G(Q_\delta)$ and $N_{G'}(Q'_{\delta'})$ determined by δ and δ' (see Section 4 below) are *basic Morita equivalent* too, which extends statement 7.7.4 in [6].

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1.2. Since Jeremy Rickard's thesis [8], it has been understood the interest in considering not only the equivalence between the categories of kGb- and $kG'b'$-modules, but more generally the equivalence between their corresponding *derived categories*. Thus, in [6, Section 19] are also considered the *basic Rickard equivalences* which extend both, the *splendid equivalences* previously introduced by Rickard [9] and the *basic Morita equivalences* mentioned above. In all these cases, the word *"basic"* means the existence of suitable *"stable bases"* which originally implies the same kind of equivalence at the level of the so-called *Brauer correspondents* in the centralizers [6, Theorem 19.11].

1.3. Our main tool in [7] in order to prove that *basic Morita equivalences* are inherited by the normalizers — more generally, by the *K-normalizers* — of the corresponding *Brauer correspondents* was the so-called *extended Brauer construction* (see Section 3 below). In this paper, we extend this result to the *basic Rickard equivalences* and, as a matter of fact, our proof only depends on the *extended Brauer construction* again and on the machinery already developed in [6]. A previous result by Andrei Marcus [1] on the same direction only covers the case where K is a p'-group — actually, Marcus' effort amounts to guessing that the equivalence for the centralizers *already forces* the equivalence for the normalizers, which it is not our approach.

1.4. Let us briefly recall the notation from [6] that we need here. In the point of view of [6], the *differential complexes* are considered as $\mathfrak{D}$*-modules* where, denoting by $\mathfrak{F}$ the commutative k-algebra of all the k-valued functions on the set $\mathbf{Z}$ of all rational integers, $\mathfrak{D}$ is the k-algebra containing $\mathfrak{F}$ as a unitary k-subalgebra and an element d such that

$$\mathfrak{D} = \mathfrak{F} \oplus \mathfrak{F}d, \qquad d^2 = 0 \quad \text{and} \quad df = \mathrm{sh}(f)d \neq 0 \quad \text{for any } f \in \mathfrak{F} - \{0\}$$

where sh denotes the automorphism on the k-algebra $\mathfrak{F}$ mapping $f \in \mathfrak{F}$ onto the k-valued function sending $z \in \mathbf{Z}$ to $f(z+1)$; moreover, we denote by s and i_z, for any $z \in \mathbf{Z}$, the k-valued functions mapping $z' \in \mathbf{Z}$ on $(-1)^{z'}$ and $\delta_z^{z'}$ respectively. Except for all the *group algebras* over $\mathfrak{D}$, we assume that all the modules and the algebras over k are finite dimensional. If A is a k-algebra we denote by A^* the group of invertible elements of A, and by A° the opposite k-algebra. Note that we have an isomorphism t: $\mathfrak{D} \cong \mathfrak{D}^\circ$ mapping $f \in \mathfrak{F}$ on the k-valued function sending $z \in \mathbf{Z}$ to $f(-z)$, and d on sd.

1.5. A $\mathfrak{D}$*-interior algebra* is a k-algebra A endowed with a unitary k-algebra homomorphism $\mathfrak{D} \to A$. Note that the isomorphism t: $\mathfrak{D} \cong \mathfrak{D}^\circ$ then determines a $\mathfrak{D}$-interior algebra structure for A°. Moreover, we have a k-algebra homomorphism $\mathfrak{D} \to k$ mapping $f + f'd$ on $f(0)$ for any $f, f' \in \mathfrak{F}$, so that any k-algebra admits a *trivial* structure of $\mathfrak{D}$-interior algebra. If A and A' are $\mathfrak{D}$-interior algebras, the tensor product $A \otimes_k A'$ admits a $\mathfrak{D}$-interior algebra structure given by

$$f \cdot (a \otimes a') = \sum_{z,z' \in \mathbf{Z}} f(z+z') i_z \cdot a \otimes i_{z'} \cdot a',$$
$$d \cdot (a \otimes a') = d \cdot a \otimes s \cdot a' + a \otimes d \cdot a'$$

which makes sense since in the sum above all but a finite number of terms vanish [6, Proposition 9.7] and since we have $\mathrm{sh}(s) = -s$.

1.6. Let G be a finite group; recall that a kG-interior algebra is a k-algebra endowed with a unitary k-algebra homomorphism from kG. Similarly, a $\mathfrak{D}G$-interior algebra is a k-algebra A endowed with a unitary k-algebra homomorphism $\rho : \mathfrak{D}G \to A$ (but A is always finite dimensional!); for any $x \in \mathfrak{D}G$ and $a \in A$, we write $x \cdot a$ and $a \cdot x$ instead of $\rho(x)a$ and $a\rho(x)$ respectively. For any subgroup H of G, we denote by A^H the centralizer of $\rho(H)$ in A; obviously $\rho(x) \in A^H$ for any $x \in \mathfrak{D}C_G(H)$ and thus the restriction of ρ to $\mathfrak{D}C_G(H)$ induces a $\mathfrak{D}C_G(H)$-interior algebra structure on A^H. If $\varphi : L \to G$ is a group homomorphism, the composition of the corresponding k-algebra homomorphism $\mathfrak{D}L \to \mathfrak{D}G$

with $\rho:\mathfrak{D}G \to A$ defines a $\mathfrak{D}L$-interior algebra structure on A; we denote this $\mathfrak{D}L$-interior algebra by $\mathrm{Res}_\varphi(A)$; we write $\mathrm{Res}^G_H(A)$ when $L = H$ and φ is the inclusion homomorphism $H \subset G$.

1.7. Let us denote by $\mathbb{C}_0(A)$ the centralizer of the image of $\mathfrak{D}$ in A; since the images of $\mathfrak{D}$ and G centralize each other, $\mathbb{C}_0(A)$ inherits a kG-interior algebra structure and, according to the terminology in [6], the *pointed groups, their inclusions, the local pointed groups,* etc., over the $\mathfrak{D}G$-interior algebra A are nothing but the pointed groups, their inclusions, the local pointed groups, etc., over the kG-interior algebra $\mathbb{C}_0(A)$. However, if H_β is a pointed group over A, so that β is a conjugacy class of primitive idempotents in $\mathbb{C}_0(A)^H$, for any $i \in \beta$ the k-algebra $A_\beta = iAi$ inherits a $\mathfrak{D}H$-interior algebra structure mapping $y \in \mathfrak{D}H$ on $y \cdot i = i \cdot y$. For any subgroup H of G, we call *contractible* any *point* contained in the two-sided ideal

$$\mathbb{B}_0\big(A^H\big) = \mathbb{C}_0(A)^H \cap \big\{d \cdot a + a \cdot d \,\big|\, a \in A^H\big\}$$

and we set $\mathbb{H}_0(A^H) = \mathbb{C}_0(A)^H/\mathbb{B}_0(A^H)$, which still inherits a $kC_G(H)$-interior algebra structure; whenever $\mathbb{H}_0(A^G) = \{0\}$ we say that *A is contractible*. It is clear that if M is a $\mathfrak{D}G$-module then $\mathrm{End}_k(M)$ is a $\mathfrak{D}G$-interior algebra and we say that M is *contractible* whenever $\mathrm{End}_k(M)$ is so [6, Corollary 10.9]; moreover, we say that M is 0*-split* if it is $\mathfrak{D}G$-isomorphic to the direct sum of a contractible $\mathfrak{D}G$-module and a kG-module endowed with the trivial $\mathfrak{D}$-structure defined above [6, 10.12].

1.8. Our standard setting in this paper is formed by two finite groups G and G', respective blocks b of G and b' of G', and an indecomposable $\mathfrak{D}(G \times G')$-module $\ddot{M}$ associated with $b \otimes b'$ such that the restrictions of $\ddot{M}$ to $G \times \{1\}$ and to $\{1\} \times G'$ are both projective. We denote by $\ddot{M}^*$ the k-dual of $\ddot{M}$ which, *via* the isomorphism t (cf. 1.4), still has a $\mathfrak{D}(G \times G')$-module structure. Following [6, 18.2.2], we say that $\ddot{M}$ defines a *Rickard equivalence* between kGb and $kG'b'$ if, for suitable contractible $\mathfrak{D}(G \times G')$- and $\mathfrak{D}(G' \times G)$-modules C and C', we have respective $\mathfrak{D}(G \times G)$- and $\mathfrak{D}(G' \times G')$-module isomorphisms

$$\ddot{M} \otimes_{kG'} \ddot{M}^* \cong kGb \oplus C \quad \text{and} \quad \ddot{M}^* \otimes_{kG} \ddot{M} \cong kG'b' \oplus C'$$

where kGb and $kG'b'$ have the trivial $\mathfrak{D}$-interior structure defined above. We say that kGb and $kG'b'$ are *Rickard equivalent* if there exists such a $\mathfrak{D}(G \times G')$-module; in this case, note that the $\mathfrak{D}(G \times G)$- and $\mathfrak{D}(G' \times G')$-modules $\ddot{M} \otimes_{kG'} \ddot{M}^*$ and $\ddot{M}^* \otimes_{kG} \ddot{M}$ are 0-split.

1.9. By our remarks in 1.7 above, in this case we still have a *vertex* $\ddot{P}$ of $\ddot{M}$ — we consider a maximal *local pointed group* $\ddot{P}_{\ddot{\gamma}}$ over the $\mathfrak{D}(G \times G')$-interior algebra $\mathrm{End}_k(\ddot{M})$ or, equivalently, over the $k(G \times G')$-interior algebra $\mathbb{C}_0(\mathrm{End}_k(\ddot{M}))$ — and a corresponding *source* $\ddot{N}$ — the $\mathfrak{D}\ddot{P}$-module $j \cdot \ddot{M}$ for some $j \in \ddot{\gamma}$. According to Theorem 18.8 in [6], the images $P \subset G$ and $P' \subset G'$ of $\ddot{P} \subset G \times G'$ by the canonical projections $\pi : G \times G' \to G$ and $\pi' : G \times G' \to G'$ are *defect groups* of b and b' respectively; then, it is clear that $\ddot{P} \times \ddot{P}$ acts on $\mathrm{End}_k(\ddot{N})$ by left and right multiplication, and we say that the Rickard equivalence between kGb and $kG'b'$ defined by $\ddot{M}$ is *basic* whenever each one of the subgroups $\ddot{P} \times_P \ddot{P}$ and $\ddot{P} \times_{P'} \ddot{P}$ of $\ddot{P} \times \ddot{P}$ stabilizes a basis of $\mathrm{End}_k(\ddot{N})$. We say that kGb and $kG'b'$ are *basic Rickard equivalent* if there exists a $\mathfrak{D}(G \times G')$-module $\ddot{M}$ defining a *basic Rickard equivalence* between kGb and $kG'b'$.

1.10. In this situation, as in the case of the *basic Morita equivalences,* we have a group isomorphism λ: $P \cong P'$ but this time λ only induces an equivalence between the *Brauer categories* of kGb and $kG'b'$ [6, Theorem 19.7]. That is to say, recall that a *Brauer* (b, G)*-pair* (Q, f) is a pair formed by a subgroup Q of G and a block f of $C_G(Q)$ such that $f\,\mathrm{Br}_Q(b) = f$ (cf. 3.1 below); since $(kG)(Q) \cong kC_G(Q)$ (cf. 3.1 below), any local point δ of Q over kGb determines a Brauer (b, G)-pair (Q, f) and it is well-known that the inclusion between the local pointed groups over kGb induces an inclusion relation between the Brauer (b, G)-pairs [6, 2.13]; then, the *Brauer category* of kGb is formed by the Brauer (b, G)-pairs and by the homomorphisms between the groups induced by the

G-conjugation and the inclusion of (b, G)-Brauer pairs. Recall that (Q, f) is called *selfcentralizing* if Q is a defect group of f as a block of $Q \cdot C_G(Q)$ [6, 2.12]; then, Q has a unique local point δ over kGb associated with f [6, 2.12.1] and Q_δ is called *selfcentralizing* too.

1.11. Moreover recall that, if (P, e) and (P', e') are Brauer (b, G)- and (b', G')-pairs, for any subgroup Q of P and any subgroup Q' of P' there are unique Brauer (b, G)- and (b', G')-pairs fulfilling [6, 2.13.2]

$$(Q, f) \subset (P, e) \quad \text{and} \quad (Q', f') \subset (P', e')$$

and that the corresponding *full subcategories* over all these objects are equivalent to the respective *Brauer categories* of kGb and $kG'b'$. Then, Theorem 19.7 in [6] states that, if kGb and $kG'b'$ are *basic Rickard equivalent*, there is a group isomorphism $\lambda\colon\ P \cong P'$ such that the correspondence mapping any Brauer (b, G)-pair $(Q, f) \subset (P, e)$ on the unique Brauer (b', G')-pair $(\lambda(Q), f^\lambda) \subset (P', e')$ induces an equivalence between the *Brauer categories* of kGb and $kG'b'$. Finally, if Q is a subgroup of P and K a subgroup of $\mathrm{Aut}(Q)$, let us denote by $N_G^K(Q)$ the converse image of K in $N_G(Q)$ — called K-*normalizer* of Q in G. We are ready to state our main result.

Theorem 1.12. *Assume that kGb and $kG'b'$ are basic Rickard equivalent. Let (P, e) and (P', e') be respective maximal Brauer (b, G)- and (b', G')-pairs. Then, there is a group isomorphism $\lambda\colon\ P \cong P'$ such that it induces an equivalence between the Brauer categories of kGb and $kG'b'$ and that, for any Brauer (b, G)-pair (Q, f) contained in (P, e) and any subgroup K of the image of $N_G(Q, f)$ in* $\mathrm{Aut}(Q)$, *the block algebras $kN_G^K(Q)f$ and $kN_G^{K'}(Q')f'$ are basic Rickard equivalent, where $Q' = \lambda(Q)$, $f' = f^\lambda$, and K' denotes the image of K in* $\mathrm{Aut}(Q')$.

2. Notation and quoted results

2.1. Let G be a finite group and A a $\mathfrak{O}G$-interior algebra; as we mention in 1.7 above, a *pointed group H_β* over A is nothing but a pointed group over the kG-interior algebra $\mathbb{C}_0(A)$, namely β is a conjugacy class of primitive idempotents in $\mathbb{C}_0(A)^H = \mathbb{C}_0(A^H)$; similarly, a pointed group P_γ over A *is contained* in H_β if $P \subset H$ and for any $i \in \beta$ there is $j \in \gamma$ such that $ij = j = ji$; we say that the point γ is *local* and that P_γ is a *local pointed group* if $\gamma \not\subset \sum_Q \mathbb{C}_0(A)_Q^P$ when Q runs over the set of proper subgroups of P.

2.2. A *homomorphism* $f : A \to A'$ between $\mathfrak{O}G$-interior algebras is a not necessarily unitary k-algebra homomorphism fulfilling $f(x \cdot a \cdot y) = x \cdot f(a) \cdot y$ for any $x, y \in \mathfrak{O}$ and any $a \in A$; then, we say that f is an *embedding* whenever $\mathrm{Ker}(f) = \{0\}$ and $\mathrm{Im}(f) = f(1) \cdot A' \cdot f(1)$. Note that any element $a \in \mathbb{C}_0(A^G)^*$ induces by conjugation a $\mathfrak{O}G$-interior algebra automorphism $\mathrm{int}(a)\colon\ A \cong A$ mapping $c \in A$ on aca^{-1}; we set

$$\tilde{f} = \bigl\{\mathrm{int}(a') \circ f \bigm| a' \in \mathbb{C}_0\bigl((A')^G\bigr)^*\bigr\}$$

and call it the *exomorphism* of $\mathfrak{O}G$-interior algebras determined by f. For instance, for any pointed group H_β over A and any $i \in \beta$, the inclusion $iAi \subset A$ determines a canonical exoembedding $\tilde{f}_\beta : A_\beta \to \mathrm{Res}_H^G(A)$ which, up to a unique $\mathfrak{O}H$-interior algebra exoisomorphism, does not depend on the choice of i.

2.3. Let H be a subgroup of G and B a $\mathfrak{O}H$-interior algebra; the *induced algebra*

$$\mathrm{Ind}_H^G(B) = kG \otimes_{kH} B \otimes_{kH} kG$$

where, for any $x, y, x', y' \in H$ and any $b, b' \in B$, the product is defined by

$$(x \otimes b \otimes y)(x' \otimes b' \otimes y') = \begin{cases} x \otimes b \cdot yx' \cdot b' \otimes y' & \text{if } yx' \in H, \\ 0 & \text{otherwise} \end{cases}$$

clearly admits a $\mathfrak{O}G$-interior structure mapping $z \in \mathfrak{O}G$ onto $\sum_y zy \otimes 1 \otimes y^{-1}$ where $y \in G$ runs over a set of representatives for G/H; let us denote by

$$d_H^G(B) : B \to \mathrm{Res}_H^G\big(\mathrm{Ind}_H^G(B)\big) \tag{2.3.1}$$

the $\mathfrak{O}H$-interior algebra embedding mapping $b \in B$ onto $1 \otimes b \otimes 1$. In order to formulate the main feature of the induced algebras, it is handy to replace A by its *Higman envelope* [6, 14.6]; namely, it follows from Proposition 14.7 and Corollary 14.11 in [6] that we may assume that

2.3.2 *For any pointed group H_β over A there is a unique $\mathfrak{O}G$-interior algebra exoembedding*

$$\tilde{h}_\beta : \mathrm{Ind}_H^G(A_\beta) \to \mathrm{Res}_H^G(A)$$

fulfilling $\mathrm{Res}_H^G(\tilde{h}_\beta) \circ \tilde{d}_H^G(A_\beta) = \tilde{f}_\beta$.

Then, Theorem 14.9 in [6] states.

Theorem 2.4. *For any pointed groups H_β and L_ε over A fulfilling $L \subset H$, the following two conditions are equivalent*:

2.4.1 $\beta \subset \mathrm{Tr}_L^H(\mathbb{C}_0(A^L) \cdot \varepsilon \cdot \mathbb{C}_0(A^L))$.
2.4.2 *There is a $\mathfrak{O}H$-exoembedding $\tilde{h}_\beta^\varepsilon : A_\beta \to \mathrm{Ind}_L^H(A_\varepsilon)$ fulfilling*

$$\tilde{h}_\varepsilon \circ \mathrm{Ind}_H^G\big(\tilde{h}_\beta^\varepsilon\big) = \tilde{h}_\beta.$$

Moreover, in this case the $\mathfrak{O}H$-exoembedding $\tilde{h}_\beta^\varepsilon$ is unique.

2.5. But, we also need the *noninjective induction* introduced in [6]; let us recall its definition. Let $\bar{G}$ be another finite group and $\rho : G \to \bar{G}$ be a surjective group homomorphism with a kernel W; the tensor product $k \otimes_{kW} A$ obviously admits a right A-module structure

$$(k \otimes_{kW} A) \times A \to k \otimes_{kW} A.$$

Since we have the equality $(1 \otimes a) \cdot b = (1 \otimes a) \cdot (x \cdot b)$ for any $a \in A$ such that $1 \otimes a \in (k \otimes_{kW} A)^W$, any $b \in A$ and any $x \in W$, the restriction to $(k \otimes_{kW} A)^W \times A$ of the above map factorizes throughout the maps

$$(k \otimes_{kW} A)^W \times A \to (k \otimes_{kW} A)^W \times (k \otimes_{kW} A),$$
$$(k \otimes_{kW} A)^W \times (k \otimes_{kW} A) \to k \otimes_{kW} A$$

and therefore the restriction of the latter to $(k \otimes_{kW} A)^W \times (k \otimes_{kW} A)^W$ induces a product on $(k \otimes_{kW} A)^W$

$$(k \otimes_{kW} A)^W \times (k \otimes_{kW} A)^W \to (k \otimes_{kW} A)^W.$$

With this product and the homomorphism $\mathfrak{O}\bar{G} \to (k \otimes_{kW} A)^W$ sending $z \in \mathfrak{O}$ to $1 \otimes (z \cdot 1_A)$ and $\bar{x} \in \bar{G}$ to $1 \otimes (x \cdot 1_A)$ where x is a lifting of $\bar{x}$ to G, $(k \otimes_{kW} A)^W$ becomes a $\mathfrak{O}\bar{G}$-interior algebra [6, 3.2.5] — noted $\mathrm{Ind}_\varphi(A)$.

2.6. A significant difference from the ordinary induction is that we have no canonical homomorphism from A, but only from the *normalizer* of W in A. Precisely, for any $\sigma \in \mathrm{Aut}(W)$, let us denote by $\Delta_\sigma : W \to W \times W$ the σ-twisted diagonal homomorphism mapping $w \in W$ on $(\sigma(w), w)$ and set

$$N_A(W) = \bigoplus_{\sigma \in \mathrm{Aut}(W)} A^{\Delta_\sigma(W)}$$

which admits a $\mathfrak{D}G$-interior algebra structure (see [7, Section 3] and Section 3 below); moreover, we have an evident unitary $\mathfrak{D}G$-interior algebra homomorphism $N_A(W) \to A$ which, as it is not difficult to prove, is injective whenever A is projective as $k(W \times W)$-module (and then $N_A(W)$ coincides with the definition in [6, 2.3.3]). Then, we denote by [6, 3.4.2]

$$d_\rho(A) : N_A(W) \to \mathrm{Res}_\rho\big(\mathrm{Ind}_\rho(A)\big)$$

the homomorphism mapping $a \in A^{\Delta_\sigma(W)}$ on $1 \otimes a$ for any $\sigma \in \mathrm{Aut}(W)$. More generally, for any group homomorphism $\varphi : G \to G'$, setting $\bar{G} = \varphi(G)$ and denoting by $\rho : G \to \bar{G}$ the group homomorphism determined by φ, we still set

$$\mathrm{Ind}_\varphi(A) = \mathrm{Ind}_{\bar{G}}^{G'}\big(\mathrm{Ind}_\rho(A)\big) \quad \text{and} \quad d_\varphi(A) = d_{\bar{G}}^{G'}\big(\mathrm{Ind}_\rho(A)\big) \circ d_\rho(A).$$

2.7. Let us come back to our standard setting (cf. 1.8) and respectively denote by $\pi_{\ddot{P}} : \ddot{P} \to G$ and $\pi'_{\ddot{P}} : \ddot{P} \to G'$ the restrictions to $\ddot{P}$ of π and π'. As in [6, 16.1], it follows from Proposition 14.7 and Corollary 14.12 in [6] that we may replace $\ddot{N}$ by a bigger $\mathfrak{D}\ddot{P}$-module $\ddot{N}'$ — with its restrictions to $\mathrm{Ker}(\pi_{\ddot{P}})$ and $\mathrm{Ker}(\pi'_{\ddot{P}})$ always projective, and containing $\ddot{N}$ as a direct summand — in such a way that $\ddot{S} = \mathrm{End}_k(\ddot{N}')$ becomes a $\mathfrak{D}\ddot{P}$-interior algebra which coincides with its *Higman envelope* (cf. 2.3.2), and admits $\ddot{P} \times_P \ddot{P}$- and $\ddot{P} \times_{P'} \ddot{P}$-stable bases whenever $\mathrm{End}_k(\ddot{N})$ does; then, setting

$$\ddot{A} = \mathrm{Ind}_{\ddot{P}}^{G \times G'}(\ddot{S}), \qquad \hat{A} = (\ddot{A})^{1 \times G'} \quad \text{and} \quad \hat{A}' = (\ddot{A})^{G \times 1}$$

we already know that all these interior algebras coincide with their *Higman envelopes* [6, 16.1.3] and that we have $\mathfrak{D}G$- and $\mathfrak{D}G'$-interior algebra isomorphisms [6, 16.1.2]

$$\hat{A} \cong \mathrm{Ind}_{\pi_{\ddot{P}}}\big(\ddot{S} \otimes_k \mathrm{Res}_{\pi'_{\ddot{P}}}(kG')\big) \quad \text{and} \quad \hat{A}' \cong \mathrm{Ind}_{\pi'_{\ddot{P}}}\big(\ddot{S} \otimes_k \mathrm{Res}_{\pi_{\ddot{P}}}(kG)\big).$$

Then, Proposition 18.4 in [6] states.

Proposition 2.8. *With the notation above, $\ddot{M}$ defines a Rickard equivalence between kGb and $kG'b'$ if and only if there are points $\hat{\alpha}$ of G over $\hat{A}$ and $\hat{\alpha}'$ of G' over $\hat{A}'$ such that $\hat{A}_{\hat{\alpha}}$ and $\hat{A}'_{\hat{\alpha}'}$ respectively considered as $\mathfrak{D}(G \times G)$- and $\mathfrak{D}(G' \times G')$-modules are 0-split and the structural homomorphisms $kGb \to \hat{A}_{\hat{\alpha}}$ and $kG'b' \to \hat{A}'_{\hat{\alpha}'}$ induce kG- and kG'-interior algebra isomorphisms*

$$kGb \cong \mathbb{H}_0(\hat{A}_{\hat{\alpha}}) \quad \textit{and} \quad kG'b' \cong \mathbb{H}_0\big(\hat{A}'_{\hat{\alpha}'}\big).$$

In this case, the structural homomorphisms induce bijections between the set of pointed groups over kGb and $kG'b'$, and the corresponding sets of noncontractible points over $\hat{A}_{\hat{\alpha}}$ and $\hat{A}'_{\hat{\alpha}'}$, preserving inclusion and localness.

2.9. In [6] the main tool to relate both sides in a *Rickard equivalence* are the so-called *local tracing triples*. In the above setting, a *local tracing triple* over $\hat{A}$, $\ddot{A}$ and $kG'b'$ is a triple of local pointed groups $Q_{\hat{\delta}}$ on $\hat{A}$, $\ddot{Q}_{\ddot{\delta}}$ and $Q'_{\delta'}$ on $kG'b'$, such that $\pi(\ddot{Q}) = Q$ and $\pi'(\ddot{Q}) = Q'$, and that, denoting by $\tau : \ddot{Q} \to Q$ and $\tau' : \ddot{Q} \to Q'$ the corresponding group homomorphisms, there is a (unique) $\mathfrak{D}Q$-interior algebra exoembedding

$$\tilde{h}_{\hat{\delta}}^{\ddot{\delta},\delta'} : \hat{A}_{\hat{\delta}} \to \mathrm{Ind}_\tau\left(\ddot{A}_{\ddot{\delta}} \otimes_k \mathrm{Res}_{\tau'}(kG')_{\delta'}\right) \tag{2.9.1}$$

in such a way that the following diagram is commutative [6, 16.5.2]

$$\begin{tikzcd}
\mathrm{Res}^{G\times G'}_{Q\times G'}(\ddot{A}) & \mathrm{Ind}^{Q\times G'}_{\ddot{Q}}(\ddot{A}_{\ddot{\delta}}) \arrow[l, "\tilde{h}_{\ddot{\delta}}"'] & \\
\cup & \cup & \\
\mathrm{Res}^G_Q(\hat{A}) & \mathrm{Ind}^{Q\times G'}_{\ddot{Q}}(\ddot{A}_{\ddot{\delta}})^{1\times G'} \cong \mathrm{Ind}_\tau(\ddot{A}_{\ddot{\delta}} \otimes_k \mathrm{Res}_{\pi'_{\ddot{Q}}}(kG')) & \\
\hat{A}_{\hat{\delta}} \arrow[u, "\tilde{f}_{\hat{\delta}}"] \arrow[r, "\tilde{h}_{\hat{\delta}}^{\ddot{\delta},\delta'}"] & \mathrm{Ind}_\tau(\ddot{A}_{\ddot{\delta}} \otimes_k \mathrm{Res}_{\tau'}(kG')_{\delta'}) \arrow[u, "\mathrm{Ind}_\tau(\widetilde{\mathrm{id}}\otimes_k \mathrm{Res}_{\tau'}(\tilde{f}_{\delta'}))"'] &
\end{tikzcd}$$

explicitly, $\tilde{h}_{\ddot{\delta}}$ induces an exoembedding from $\mathrm{Ind}^{Q\times G'}_{\ddot{Q}}(\ddot{A}_{\ddot{\delta}})^{1\times G'}$ to $\mathrm{Res}^G_Q(\hat{A})$ and therefore, by composition, we obtain an exoembedding

$$\tilde{g}_{\ddot{\delta},\delta'} : \mathrm{Ind}_\tau\left(\ddot{A}_{\ddot{\delta}} \otimes_k \mathrm{Res}_{\tau'}(kG')_{\delta'}\right) \to \mathrm{Res}^G_Q(\hat{A})$$

so that $\tilde{h}_{\hat{\delta}}^{\ddot{\delta},\delta'}$ fulfills $\tilde{g}_{\ddot{\delta},\delta'} \circ \tilde{h}_{\hat{\delta}}^{\ddot{\delta},\delta'} = \tilde{f}_{\hat{\delta}}$; thus, the existence of $\tilde{h}_{\hat{\delta}}^{\ddot{\delta},\delta'}$ is equivalent to the equality

$$f_{\hat{\delta}}(1) g_{\ddot{\delta},\delta'}(1) = f_{\hat{\delta}}(1) = g_{\ddot{\delta},\delta'}(1) f_{\hat{\delta}}(1) \tag{2.9.2}$$

for a suitable choice of representatives $f_{\hat{\delta}}$ and $g_{\ddot{\delta},\delta'}$.

2.10. Moreover, we consider an *inclusion relation* between the local tracing triples refining the corresponding inclusions between the respective local pointed groups; we say that a local tracing triple $(R_{\hat{\varepsilon}}, \ddot{R}_{\ddot{\varepsilon}}, R'_{\hat{\varepsilon}'})$ *is contained in* $(Q_{\hat{\delta}}, \ddot{Q}_{\ddot{\delta}}, Q'_{\delta'})$ and we write

$$\left(R_{\hat{\varepsilon}}, \ddot{R}_{\ddot{\varepsilon}}, R'_{\varepsilon'}\right) \subset \left(Q_{\hat{\delta}}, \ddot{Q}_{\ddot{\delta}}, Q'_{\delta'}\right)$$

if we have $R_{\hat{\varepsilon}} \subset Q_{\hat{\delta}}$, $\ddot{R}_{\ddot{\varepsilon}} \subset \ddot{Q}_{\ddot{\delta}}$ and $R'_{\varepsilon} \subset Q'_{\delta'}$ and there is a $\mathfrak{D}R$-interior algebra exoembedding

$$\tilde{g}_{\ddot{\varepsilon},\varepsilon'}^{\ddot{\delta},\delta'} : \mathrm{Ind}_\rho\left(\ddot{A}_{\ddot{\varepsilon}} \otimes_k \mathrm{Res}_{\rho'}(kG')_{\varepsilon'}\right) \to \mathrm{Res}^Q_R\left(\mathrm{Ind}_\tau\left(\ddot{A}_{\ddot{\delta}} \otimes_k \mathrm{Res}_{\tau'}(kG')_{\delta'}\right)\right)$$

fulfilling

$$\tilde{g}_{\ddot{\varepsilon},\varepsilon'} = \mathrm{Res}^Q_R(\tilde{g}_{\ddot{\delta},\delta'}) \circ \tilde{g}_{\ddot{\varepsilon},\varepsilon'}^{\ddot{\delta},\delta'}$$

where $\rho : \ddot{R} \to R$ and $\rho' : \ddot{R} \to R'$ are the respective restrictions of τ and τ'. Then, Theorem 16.15 in [6] guarantees that, for any local pointed group R_ε contained in Q_δ there is a local tracing triple $(R_{\hat{\varepsilon}}, \ddot{R}_{\ddot{\varepsilon}}, R'_{\hat{\varepsilon}'})$ contained in $(Q_{\hat{\delta}}, \ddot{Q}_{\ddot{\delta}}, Q'_{\delta'})$.

2.11. Let us come to the "basic" situations. We say that a local tracing triple $(Q_{\hat\delta}, \ddot Q_{\ddot\delta}, Q'_{\delta'})$ over $\hat A$, $\ddot A$ and $kG'b'$ is *basic* [6, 17.1] whenever $\ddot A_{\ddot\delta}$ is a projective $k(\mathrm{Ker}(\tau) \times \mathrm{Ker}(\tau))$-module, the subgroup $\ddot Q \times_Q \ddot Q$ of $\ddot Q \times \ddot Q$ stabilizes a basis of $\ddot A_{\ddot\delta}$ and $\hat A_{\hat\delta}$ considered as a $\mathfrak{D}(Q \times Q)$-module is 0-split (actually, the last condition is stronger than in [6, 17.1]). In this case, τ admits a section and, for any section $\mu : Q \to \ddot Q$ of τ, let us set $Q^\mu = \mu(Q)$ and $Q^{\mu'} = \tau(Q^\mu)$, and let us denote by

$$Z(Q) \xleftarrow{\tau_\mu} C_{\ddot Q}\big(Q^\mu\big) \xrightarrow{\tau_{\mu'}} Z\big(Q^{\mu'}\big)$$

the respective restrictions of τ and τ'; then, it is clear that the $\mathfrak{D}\ddot Q$-interior algebra $\ddot A_{\ddot\delta} \otimes_k \mathrm{Res}_{\tau'}(kG')_{\delta'}$ is also a projective $k(\mathrm{Ker}(\tau) \times \mathrm{Ker}(\tau))$-module and that the subgroup $\ddot Q \times_Q \ddot Q$ of $\ddot Q \times \ddot Q$ stabilizes a basis on it; hence, according to Theorem 13.9 in [6] applied to this $\mathfrak{D}\ddot Q$-interior algebra and to Q, for any section μ of τ we have a $\mathfrak{D}Z(Q)$-interior algebra embedding

$$\begin{array}{c}\mathrm{Ind}_\tau(\ddot A_{\ddot\delta} \otimes_k \mathrm{Res}_{\tau'}(kG')_{\delta'})(Q) \\ \Big\uparrow{\scriptstyle e_\mu^{\ddot\delta,\delta'}} \\ \mathrm{Ind}_{\tau_\mu}(\ddot A_{\ddot\delta}(Q^\mu) \otimes_k \mathrm{Res}_{\tau'_\mu}((kG')_{\delta'}(Q^{\mu'})))\end{array}$$

and the set of idempotents $e_\mu^{\ddot\delta,\delta'}(1)$ form an orthogonal decomposition of the unity in the top k-algebra. We say that a section μ of τ is a *section* of the basic local tracing triple $(Q_{\hat\delta}, \ddot Q_{\ddot\delta}, Q'_{\delta'})$ if we have a factorization [6, 17.4]

$$\begin{tikzcd} \hat A_{\hat\delta}(Q) \arrow[r, "\tilde h_{\hat\delta}^{\ddot\delta,\delta'}(Q)"] \arrow[dr, "\tilde h_{\hat\delta}^{\mu}"'] & \mathrm{Ind}_\tau(\ddot A_{\ddot\delta} \otimes_k \mathrm{Res}_{\tau'}(kG')_{\delta'})(Q) \\ & \mathrm{Ind}_{\tau_\mu}(\ddot A_{\ddot\delta}(Q^\mu) \otimes_k \mathrm{Res}_{\tau'_\mu}((kG')_{\delta'}(Q^{\mu'}))) \arrow[u, "\tilde e_\mu^{\ddot\delta,\delta'}"'] \end{tikzcd}$$

where we set $Q^\mu = \mu(Q)$ and $Q^{\mu'} = \tau'(Q^\mu)$, and denote by τ_μ and τ'_μ the respective restrictions of τ and τ'.

2.12. In this case, $\tilde h_{\hat\delta}^\mu$ is a unique exoembedding [6, 2.11.4] and it follows from the equality $\tilde g_{\ddot\delta,\delta'} \circ \tilde h_{\hat\delta}^{\ddot\delta,\delta'} = \tilde f_{\hat\delta}$ that

$$\tilde g_{\ddot\delta,\delta'}(Q) \circ \tilde e_\mu^{\ddot\delta,\delta'} \circ \tilde h_{\hat\delta}^\mu = \tilde g_{\ddot\delta,\delta'}(Q) \circ \tilde h_{\hat\delta}^{\ddot\delta,\delta'}(Q) = \tilde f_{\hat\delta}(Q).$$

Conversely, assume that $Q_{\hat\delta}$, $\ddot Q_{\ddot\delta}$ and $Q'_{\delta'}$ are respective local pointed groups on $\hat A$, $\ddot A$ and kG' such that $\ddot A_{\ddot\delta}$ is a projective $k(\mathrm{Ker}(\tau) \times \mathrm{Ker}(\tau))$-module, that the subgroup $\ddot Q \times_Q \ddot Q$ of $\ddot Q \times \ddot Q$ stabilizes a basis of $\ddot A_{\ddot\delta}$, that $\hat A_{\hat\delta}$ is 0-split considered as a $\mathfrak{D}(Q \times Q)$-module and that, for some section μ of τ, there is a $\mathfrak{D}Z(Q)$-interior algebra exoembedding

$$\tilde h : \hat A_{\hat\delta}(Q) \to \mathrm{Ind}_{\tau_\mu}\big(\ddot A_{\ddot\delta}\big(Q^\mu\big) \otimes_k \mathrm{Res}_{\tau'_\mu}\big((kG')_{\delta'}\big(Q^{\mu'}\big)\big)\big)$$

fulfilling

$$\tilde{g}_{\ddot{\delta},\delta'}(Q) \circ \tilde{e}_{\mu}^{\ddot{\delta},\delta'} \circ \tilde{h} = \tilde{f}_{\hat{\delta}}(Q);$$

then $(Q_{\hat{\delta}}, \ddot{Q}_{\ddot{\delta}}, Q'_{\delta'})$ is a basic local tracing triple, since the equality (2.9.2) can be easily obtained from this one. Moreover, it follows from Corollary 17.9 in [6] that

2.12.1 *if μ and μ' are sections of $(Q_{\hat{\delta}}, \ddot{Q}_{\ddot{\delta}}, Q'_{\delta'})$, there is $x' \in G'$ such that $(1, x')$ normalizes $\ddot{Q}$ and that, denoting by $\varphi_{(1,x')}$ the action of $(1, x')$ on $\ddot{Q}$, we have $\mu' = \varphi_{(1,x')} \circ \mu$.*

2.13. For any subgroup R of Q, it is clear that the restriction of a section μ of $\tau : \ddot{Q} \to Q$ to a subgroup $\ddot{R}$ of $\tau^{-1}(R)$ such that $\mu(R) \subset \ddot{R}$ is also a section of the restriction $\rho : \ddot{R} \to R$ of τ; obviously, $\ddot{A}_{\ddot{\delta}}$ is still a projective $k(\mathrm{Ker}(\rho) \times \mathrm{Ker}(\rho))$-module and the subgroup $\ddot{R} \times_R \ddot{R}$ of $\ddot{R} \times \ddot{R}$ stabilizes a basis of $\ddot{A}_{\ddot{\delta}}$; similarly, if R has a local point $\hat{\varepsilon}$ on $\hat{A}$ such that $R_{\hat{\varepsilon}} \subset Q_{\hat{\delta}}$, $\hat{A}_{\hat{\varepsilon}}$ considered as a $\mathfrak{D}(R \times R)$-module is 0-split. In particular, a local tracing triple $(R_{\hat{\varepsilon}}, \ddot{R}_{\ddot{\varepsilon}}, R'_{\varepsilon'})$ on $\hat{A}$, $\ddot{A}$ and kG' contained in $(Q_{\hat{\delta}}, \ddot{Q}_{\ddot{\delta}}, Q'_{\delta'})$ is basic too and we are interested in choosing it in such a way that the restriction ν to R of a section μ of $(Q_{\hat{\delta}}, \ddot{Q}_{\ddot{\delta}}, Q'_{\delta'})$ remains a section of $(R_{\hat{\varepsilon}}, \ddot{R}_{\ddot{\varepsilon}}, R'_{\varepsilon'})$.

Theorem 2.14. *Let $(Q_{\hat{\delta}}, \ddot{Q}_{\ddot{\delta}}, Q'_{\delta'})$ be a basic local tracing triple on $\hat{A}$, $\ddot{A}$ and kG'. For any local pointed group $R_{\hat{\varepsilon}}$ on $\hat{A}$ contained in $Q_{\hat{\delta}}$, there is a basic local tracing triple $(R_{\hat{\varepsilon}}, \ddot{R}_{\ddot{\varepsilon}}, R'_{\varepsilon'})$ contained in $(Q_{\hat{\delta}}, \ddot{Q}_{\ddot{\delta}}, Q'_{\delta'})$ such that the restriction ν of μ to R remains a section of $(R_{\hat{\varepsilon}}, \ddot{R}_{\ddot{\varepsilon}}, R'_{\varepsilon'})$.*

Proof. Arguing by induction on $|Q : R|$, we may assume that $R_{\hat{\varepsilon}}$ is normal in $Q_{\hat{\delta}}$. Let μ be a section of $(Q_{\hat{\delta}}, \ddot{Q}_{\ddot{\delta}}, Q'_{\delta'})$, set

$$\ddot{C}_{\ddot{\delta},\delta'} = \ddot{A}_{\ddot{\delta}} \otimes_k \mathrm{Res}_{\tau'}(kG')_{\delta'}, \qquad \ddot{T} = \tau^{-1}(R) \quad \text{and} \quad T' = \tau'(\ddot{T}),$$

and denote by $\theta : \ddot{T} \to R$ and $\theta' : \ddot{T} \to T'$ the respective restrictions of τ and τ', by $\nu : R \to \ddot{T}$ the restriction of μ, by R^{ν} the image of R in $\ddot{T}$ and by τ_{ν} the corresponding restriction of τ; it follows from Proposition 13.16 in [6] that we have the commutative diagram of $\mathfrak{D}$-algebra exoembeddings

$$\begin{tikzcd}
\hat{A}_{\hat{\delta}}(Q) \arrow[rr, "\tilde{h}_{\hat{\delta}}^{\ddot{\delta},\delta'}(Q)"] \arrow[dr, "\tilde{h}_{\hat{\delta}}^{\mu}"'] & & (\mathrm{Ind}_{\tau}(\ddot{C}_{\ddot{\delta},\delta'}))(Q) & = & (\mathrm{Ind}_{\tau}(\ddot{C}_{\ddot{\delta},\delta'}))(Q) \\
& \mathrm{Ind}_{\tau_{\mu}}(\ddot{C}_{\ddot{\delta},\delta'}(Q^{\mu})) \arrow[ur, "\tilde{e}_{\mu}^{\ddot{\delta},\delta'}"'] \arrow[drr] & & & \\
& & & (\mathrm{Ind}_{\tau_{\nu}}(\ddot{C}_{\ddot{\delta},\delta'}(R^{\nu})))(Q/R) \arrow[uur] &
\end{tikzcd}$$

Then, Theorem 13.9 in [6] applied to $\ddot{C}_{\ddot{\delta},\delta'}$ with respect to R and the corresponding *Brauer homomorphisms* [6, Lemma 7.10] determines the following commutative diagram

$$\begin{tikzcd}
\hat A_{\hat\delta}(R)^Q \arrow[rr] \arrow[dr] \arrow[dd, phantom, "\cap"] & & (\mathrm{Ind}_\tau(\ddot C_{\ddot\delta,\delta'}))(R)^Q \arrow[dd, phantom, "\cap"] \\
& \mathrm{Ind}_{\tau_\nu}(\ddot C_{\ddot\delta,\delta'}(R^\nu))^Q \arrow[ur] \arrow[dd, phantom, "\cap"] & \\
\hat A_{\hat\delta}(R) \arrow[rr, "\tilde h^{\ddot\delta,\delta'}_{\hat\delta}(R)"] \arrow[dr] & & (\mathrm{Ind}_\tau(\ddot C_{\ddot\delta,\delta'}))(R) \\
& \mathrm{Ind}_{\tau_\nu}(\ddot C_{\ddot\delta,\delta'}(R^\nu)) \arrow[ur, "\tilde e^{\ddot\delta,\delta'}_\nu"'] &
\end{tikzcd}$$

indeed, for a suitable representative h_Q of $\tilde h^{\ddot\delta,\delta'}_{\hat\delta}(Q)$, it follows from the diagram above that we have

$$h_Q(1)e^{\ddot\delta,\delta'}_\mu(1) = h_Q(1) = e^{\ddot\delta,\delta'}_\mu(1)h_Q(1)$$

which proves the existence of the left-hand embeddings.

On the other hand, it follows from Proposition 12.12 in [6] that we have a unique isomorphism of $\mathfrak{D}R$-interior algebras

$$d^Q_{R,\tau}: \mathrm{Ind}_\theta\big(\mathrm{Res}^{\ddot Q}_{\ddot T}(\ddot C_{\ddot\delta,\delta'})\big) \cong \mathrm{Res}^Q_R\big(\mathrm{Ind}_\tau(\ddot C_{\ddot\delta,\delta'})\big)$$

mapping $1\otimes a$ on $1\otimes a$ for any $a\in \ddot C_{\ddot\delta,\delta'}$ such that $\mathrm{Ker}(\tau)$ fixes $1\otimes a$ in $k\otimes_{k\,\mathrm{Ker}(\tau)}\ddot C_{\ddot\delta,\delta'}$; now, consider an orthogonal primitive idempotent decomposition $\ddot L$ of the unity element in $\mathbb{C}_\circ(\dot C_{\ddot\delta,\delta'})^{\ddot T}$, so that setting

$$\hat\ell = 1\otimes\ddot\ell \quad\text{and}\quad \bar\ell = \mathrm{Br}_R^{\mathrm{Ind}(\ddot C_{\ddot\delta,\delta'})}(\hat\ell),$$

the family $\{\bar\ell\}_{\ddot\ell\in\ddot L}$ form an orthogonal idempotent decomposition of the unity element in $(\mathrm{Ind}_\tau(\ddot C_{\ddot\delta,\delta'}))(R)$.

Considering $\hat\jmath\in\hat\varepsilon$, we can choose a representative h_R of $\tilde h^{\ddot\delta,\delta'}_{\hat\delta}(R)$ such that $h_R(\mathrm{Br}_R^{\hat A_{\hat\delta}}(\hat\jmath))$ centralizes that family and, since this idempotent is primitive, there is $\ddot\ell\in\ddot L$ fulfilling

$$h_R\big(\mathrm{Br}_R^{\hat A_{\hat\delta}}(\hat\jmath)\big)\bar\ell = h_R\big(\mathrm{Br}_R^{\hat A_{\hat\delta}}(\hat\jmath)\big) = \bar\ell h_R\big(\mathrm{Br}_R^{\hat A_{\hat\delta}}(\hat\jmath)\big);$$

let $\ddot\eta$ be the point of $\ddot T$ on $\mathbb{C}_\circ(\ddot C_{\ddot\delta,\delta'})$ determined by $\ddot\ell$; setting $\ddot C_{\ddot\eta} = (\ddot C_{\ddot\delta,\delta'})_{\ddot\eta}$, the canonical exoembedding $\tilde f^{\ddot\delta,\delta'}_{\ddot\eta}: C_{\ddot\eta}\to \mathrm{Res}^{\ddot Q}_{\ddot T}(\ddot C_{\ddot\delta,\delta'})$ determines the following commutative diagram of $\mathfrak{D}$-algebra exoembeddings

$$\begin{tikzcd}
\hat A_{\hat\delta}(R) \arrow[r, "\tilde h^{\ddot\delta,\delta'}_{\hat\delta}(R)"] & (\mathrm{Ind}_\tau(\ddot C_{\ddot\delta,\delta'}))(R) \\
\hat A_{\hat\varepsilon}(R) \arrow[u, "\tilde f^{\delta}_{\varepsilon}(R)"] \arrow[r] & (\mathrm{Ind}_\theta(\ddot C_{\ddot\eta}))(R) \arrow[u, "(\mathrm{Ind}_\theta(\tilde f^{\ddot\delta,\delta'}_{\ddot\eta}))(R)"']
\end{tikzcd} \tag{2.14.1}$$

Thus, applying again Theorem 13.9 in [6] and denoting by $\{\tilde e^{\ddot\eta}_{\bar\nu}\}_{\bar\nu\in\check{\mathcal{S}}_\theta}$ the corresponding family of exoembeddings to $(\mathrm{Ind}_\theta(\ddot C_{\ddot\eta}))(R)$, from Proposition 13.17 in [6] we get the following commutative diagram of $\mathfrak{D}$-algebra exoembeddings

$$\begin{tikzcd}
\hat{A}_{\hat\delta}(R) \arrow[rr, "\tilde{h}^{\ddot\delta,\delta'}_{\hat\delta}(R)"] \arrow[dr] & & (\mathrm{Ind}_\tau(\ddot{C}_{\ddot\delta,\delta'}))(R) \\
& \mathrm{Ind}_{\tau_\nu}(\ddot{C}_{\ddot\delta,\delta'}(R^\nu)) \arrow[ur, "\tilde{e}^{\ddot\delta,\delta'}_\nu"'] & \\
\hat{A}_{\hat\varepsilon}(R) \arrow[uu, "\tilde{f}^\delta_\varepsilon(R)"] \arrow[rr] \arrow[dr] & {} & (\mathrm{Ind}_\theta(\ddot{C}_{\ddot\eta}))(R) \arrow[uu, "(\mathrm{Ind}_\theta(\tilde{f}^{\ddot\delta,\delta'}_{\ddot\eta}))(R)"'] \\
& \mathrm{Ind}_{\theta_\nu}(\ddot{C}_{\ddot\eta}(R^\nu)) \arrow[u] \arrow[ur, "\tilde{e}^{\ddot\eta}_\nu"'] &
\end{tikzcd} \tag{2.14.2}$$

where θ_ν denotes the corresponding restriction of θ.

But, recall that any local point $\ddot\varepsilon \times \varepsilon'$ of a subgroup $\ddot{R}$ of $\ddot{Q}$ over $\ddot{C}_{\ddot\delta,\delta'}$ is given by a local point $\ddot\varepsilon$ over $\mathbb{C}_\circ(\ddot{A}_{\ddot\delta})$ and a local point ε' over $(kG')_{\delta'}$ [4, Proposition 5.6], and that, setting $R' = \tau'(\ddot{R})$ and denoting by $\rho : \ddot{R} \to R$ and $\rho' : \ddot{R} \to R'$ the respective restrictions of θ and θ', we have a commutative diagram of $\mathfrak{D}\ddot{R}$-algebra exoembeddings [6, 14.2]

$$\begin{tikzcd}
& \mathrm{Res}^{\ddot{Q}}_{\ddot{R}}(\ddot{C}_{\ddot\delta,\delta'}) & \\
(\ddot{C}_{\ddot\delta,\delta'})_{\ddot\varepsilon\times\varepsilon'} \arrow[ur, "\tilde{f}^{\ddot\delta,\delta'}_{\ddot\varepsilon,\varepsilon'}"] \arrow[rr] & & C_{\ddot\varepsilon,\varepsilon'} = \ddot{A}_{\ddot\varepsilon} \otimes \mathrm{Res}_{\rho'}(kG')_{\varepsilon'} \arrow[ul, "\tilde{f}^{\ddot\delta}_{\ddot\varepsilon}\otimes\tilde{f}^{\delta'}_{\varepsilon'}"']
\end{tikzcd} \tag{2.14.3}$$

moreover, choosing a defect pointed group $\ddot{R}_{\ddot\varepsilon}$ of $\ddot{T}_{\ddot\eta}$, the *Green Indecomposability Theorem* suitably generalized [6, 2.12.2] guarantees that

$$\ddot{C}_{\ddot\eta} \cong \mathrm{Ind}^{\ddot{T}}_{\ddot{R}}\big((\ddot{C}_{\ddot\delta,\delta'})_{\ddot\varepsilon\times\varepsilon'}\big). \tag{2.14.4}$$

Then, this isomorphism and the commutative diagrams (2.14.1) and (2.14.3) provide the bottom part of the following commutative diagram of $\mathfrak{D}$-algebra exoembeddings

$$\begin{tikzcd}
& \hat{A}(R) & \\
\hat{A}_{\hat\delta}(R) \arrow[ur, "\tilde{f}_{\hat\delta}(R)"] \arrow[rr, "\tilde{h}^{\ddot\delta,\delta'}_{\hat\delta}(R)"] & & (\mathrm{Ind}_\tau(\ddot{C}_{\ddot\delta,\delta'}))(R) \arrow[ul, "\tilde{g}_{\ddot\delta,\delta'}(R)"'] \\
\hat{A}_{\hat\varepsilon}(R) \arrow[u, "\tilde{f}^\delta_\varepsilon(R)"] \arrow[rr, "\tilde{h}^{\ddot\varepsilon,\varepsilon'}_{\hat\varepsilon}(R)"] & & (\mathrm{Ind}_\rho(\ddot{C}_{\ddot\varepsilon,\varepsilon'}))(R) \arrow[u, "\tilde{g}^{\ddot\delta,\delta'}_{\ddot\varepsilon,\varepsilon'}(R)"']
\end{tikzcd}$$

Firstly note that the bottom embedding proves that R has a local point over $\mathrm{Ind}_\rho(\ddot{C}_{\ddot\varepsilon,\varepsilon'})$ and therefore we necessarily have $\rho(\ddot{R}) = R$; thus, this commutative diagram proves that $(R_{\hat\varepsilon}, \ddot{R}_{\ddot\varepsilon}, R'_{\varepsilon'})$ is a local tracing triple on $\hat{A}$, $\ddot{A}$ and kG' (cf. 2.9), and that it is contained in $(Q_{\hat\delta}, \ddot{Q}_{\ddot\delta}, Q'_{\delta'})$ (cf. 2.10). Moreover, we have [6, Corollary 12.7]

$$\mathrm{Ind}_\theta(\ddot{C}_{\ddot\eta}) \cong \mathrm{Ind}_\theta\big(\mathrm{Ind}_{\ddot{R}}^{\ddot{T}}\big((\ddot{C}_{\ddot\delta,\delta'})_{\ddot\varepsilon\times\varepsilon'}\big)\big) \cong \mathrm{Ind}_\rho\big((\ddot{C}_{\ddot\delta,\delta'})_{\ddot\varepsilon\times\varepsilon'}\big);$$

but, applying again Theorem 13.9 in [6] and denoting by $\{\tilde{e}_{\bar\omega}^{\ddot\varepsilon\times\varepsilon'}\}_{\bar\omega\in\check{\mathcal{S}}_\rho}$ the corresponding family of pairwise of exoembeddings to $(\mathrm{Ind}_\rho((\ddot{C}_{\ddot\delta,\delta'})_{\ddot\varepsilon\times\varepsilon'}))(R)$, we get two families of embeddings to this $\mathfrak{D}$-algebra, namely $\{\tilde{e}_{\bar\omega}^{\ddot\varepsilon\times\varepsilon'}\}_{\bar\omega\in\check{\mathcal{S}}_\rho}$ and the composition of $\{\tilde{e}_{\bar\nu}^{\ddot\eta}\}_{\bar\nu\in\check{\mathcal{S}}_\theta}$ with the isomorphism above, both providing orthogonal idempotent decompositions of the unity element; note that $\tilde{e}_{\bar\nu}^{\ddot\eta}(1) \neq 0$ forces $\ddot{C}_{\ddot\eta}(R^{\bar\nu}) \neq \{0\}$, so that a $\ddot{T}$-conjugate of $R^{\bar\nu}$ is contained in $\ddot{R}$ [6, 2.9.5]. Then, it follows from the uniqueness part of this theorem that, for some section $\omega \in \check{\mathcal{S}}_\rho$, $\tilde{e}_\nu^{\ddot\eta}$ coincides with $\tilde{e}_{\bar\omega}^{\ddot\varepsilon\times\varepsilon'}$, so that a $\mathrm{Ker}(\theta)$-conjugate of ν coincides with ω; actually, up to modifying our choice of the defect pointed group $\ddot{R}_{\ddot\varepsilon}$, we may assume that ν coincides with ω.

Finally, denoting by ρ_ν the corresponding restriction of ρ, from the bottom exoembedding in diagram (2.14.3) and from the isomorphism above we get a $\mathfrak{D}$-algebra exoembedding

$$\tilde{c} : \big(\mathrm{Ind}_\theta(\ddot{C}_{\ddot\eta})\big)(R) \to \big(\mathrm{Ind}_\rho(\ddot{C}_{\ddot\varepsilon,\varepsilon'})\big)(R)$$

and it follows from Proposition 13.17 in [6] applied to the bottom exoembedding in diagram (2.14.3) that, for a representative c of $\tilde{c}$, we have

$$c\big(e_\nu^{\ddot\eta}(1)\big)e_\nu^{\ddot\varepsilon,\varepsilon'}(1) = c\big(e_\nu^{\ddot\eta}(1)\big) = e_\nu^{\ddot\varepsilon,\varepsilon'}(1)c\big(e_\nu^{\ddot\eta}(1)\big).$$

At this point, from diagram (2.14.2) above and from this equality we get the following commutative diagram of $\mathfrak{D}$-algebra exoembeddings

$$\begin{tikzcd}
\hat{A}_{\hat\delta}(R) \arrow[rr, "\tilde{h}_{\hat\delta}^{\ddot\delta,\delta'}(R)"] \arrow[dr] & & (\mathrm{Ind}_\tau(\ddot{C}_{\ddot\delta,\delta'}))(R) \\
& \mathrm{Ind}_{\tau_\nu}(\ddot{C}_{\ddot\delta,\delta'}(R^\nu)) \arrow[ur, "\tilde{e}_\nu^{\ddot\delta,\delta'}"'] & \\
\hat{A}_{\hat\varepsilon}(R) \arrow[uu, "\tilde{f}_\varepsilon^\delta(R)"] \arrow[rr, "\tilde{h}_{\hat\varepsilon}^{\ddot\varepsilon,\varepsilon'}(R)"] \arrow[dr] & & (\mathrm{Ind}_\rho(\ddot{C}_{\ddot\varepsilon,\varepsilon'}))(R) \arrow[uu, "\tilde{g}_{\ddot\varepsilon,\varepsilon'}^{\ddot\delta,\delta'}(R)"'] \\
& \mathrm{Ind}_{\rho_\nu}(\ddot{C}_{\ddot\varepsilon,\varepsilon'}(R^\nu)) \arrow[ur, "\tilde{e}_\nu^{\ddot\varepsilon,\varepsilon'}"'] &
\end{tikzcd}$$

which proves that ν is a section of $(R_{\hat\varepsilon}, \ddot{R}_{\ddot\varepsilon}, R'_{\varepsilon'})$; indeed, choosing suitable representatives f of $\tilde{f}_\varepsilon^\delta(R)$ and g of $\tilde{g}_{\ddot\varepsilon,\varepsilon'}^{\ddot\delta,\delta'}(R)$, we may assume that the idempotents $h_R(f(1))$, $e_\nu^{\ddot\delta,\delta'}(1)$ and $g(1)$ centralize each other; then, we have

$$h_R\big(f(1)\big)e_\nu^{\ddot\delta,\delta'}(1)g(1) = h_R\big(f(1)\big)$$

and this equality proves the existence of the bottom left-hand exoembedding. We are done. □

3. The extended Brauer quotient

3.1. Here we extend to our new setting our construction in [7, Section 3]. Let G be a finite group and V be a kG-module; recall that for any subgroup P of G, we denote by V^P the k-submodule of all P-fixed elements of V, by $V(P)$ the quotient

$$V(P) = V^P \Big/ \sum_R V_R^P,$$

where R runs over the set of all proper subgroups of P and V_R^P is the image of the usual relative trace map $\mathrm{Tr}_R^P : V^R \to V^P$, and by Br_P^V the canonical surjective homomorphism $V^P \to V(P)$, which is the so-called *Brauer homomorphism* associated to P and V. Obviously, the kG-module structure on V induces $kN_G(P)$-module structures on both V^P and $V(P)$, and Br_P^V is a homomorphism of $kN_G(P)$-modules. If H is normal in G and $V = kH$, then for any p-subgroup P of G, P acts on H by conjugation and thus on kH and it is easily checked that $V(P) \cong kC_H(P)$ as $kN_G(P)$-modules; in this case, we often identify $V(P)$ with $kC_H(P)$. If V' is a p-permutation kG-module, there is a $kN_G(P)$-module isomorphism [6, Lemma 7.10]

$$\mathfrak{Br}_P^{V,V'} : (V \otimes_k V')(P) \cong V(P) \otimes_k V'(P)$$

mapping $\mathrm{Br}_P^{V\otimes_k V'}(v \otimes v')$ onto $\mathrm{Br}_P^V(v) \otimes \mathrm{Br}_P^{V'}(v')$ for any $v \in V^P$ and $v' \in V'^P$. If $f : V \to V''$ is a homomorphism of kG-modules, for any subgroup P of G and any subgroup R of P, we have $f(V^P) \subset V''^P$ and $f(V_R^P) \subset V''^P_R$, so that f induces a $kN_G(P)$-module homomorphism $f(P) : V(P) \to V''(P)$.

3.2. Let $\ddot{M}$ be a $\mathfrak{O}(G \times G)$-module, Q a p-subgroup of G and K a subgroup of the automorphism group $\mathrm{Aut}(Q)$ of Q. For any $\varphi \in K$, as above we denote by $\Delta_\varphi : Q \to Q \times Q$ the φ-twisted diagonal homomorphism mapping $u \in Q$ onto $(\varphi(u), u)$ and we set

$$N_{\ddot{M}}^\varphi(Q) = \ddot{M}^{\Delta_\varphi(Q)} \quad \text{and} \quad N_{\ddot{M}}^K(Q) = \bigoplus_{\varphi\in K} N_{\ddot{M}}^\varphi(Q).$$

It is clear that $N_{\ddot{M}}^\varphi(Q)$ inherits a $\mathfrak{O}$-module structure and, for any elements $x, y \in N_G^K(Q)$, denoting by $\bar{x}$ and $\bar{y}$ their images in $\mathrm{Aut}(Q)$, we have

$$(x, y) \cdot N_{\ddot{M}}^\varphi(Q) = N_{\ddot{M}}^{\bar{x}\circ\varphi\circ\bar{y}}(Q);$$

that is to say, the $\mathfrak{O}(G \times G)$-module structure over $\ddot{M}$ determines over $N_{\ddot{M}}^K(Q)$ a $\mathfrak{O}(N_G^K(Q) \times N_G^K(Q))$-module structure.

3.3. Similarly, for any subgroup R of Q, it is clear that the relative trace map from $\ddot{M}^{\Delta_\varphi(R)}$ to $\ddot{M}^{\Delta_\varphi(Q)}$ is a $\mathfrak{O}$-module homomorphism and that we get

$$(x, y) \cdot \ddot{M}_{\Delta_\varphi(R)}^{\Delta_\varphi(Q)} = \ddot{M}_{\Delta_{\bar{x}\circ\varphi\circ\bar{y}}(R)}^{\Delta_{\bar{x}\circ\varphi\circ\bar{y}}(Q)};$$

thus, the direct sum $\bigoplus_{\varphi\in K} \mathrm{Ker}(\mathrm{Br}_{\Delta_\varphi(Q)}^{\ddot{M}})$ is a $\mathfrak{O}(N_G^K(Q) \times N_G^K(Q))$-submodule of $N_{\ddot{M}}^K(Q)$. As in [7], in the present paper we are mainly interested in the following quotient

$$\bar{N}_{\ddot{M}}^K(Q) = N_{\ddot{M}}^K(Q) \Big/ \Big(\bigoplus_{\varphi\in K} \mathrm{Ker}\big(\mathrm{Br}_{\Delta_\varphi(Q)}^{\ddot{M}}\big)\Big) = \bigoplus_{\varphi\in K} \ddot{M}\big(\Delta_\varphi(Q)\big)$$

— called the *extended Brauer quotient* of $\ddot{M}$ at (Q, K). If $f: \ddot{M} \to \ddot{M}'$ is a $\mathfrak{D}(G \times G)$-module homomorphism, it is clear that f maps $N_{\ddot{M}}^{\varphi}(Q)$ inside $N_{\ddot{M}'}^{\varphi}(Q)$ and induces a $\mathfrak{D}(N_G^K(Q) \times N_G^K(Q))$-module homomorphism $N_f^K(Q): N_{\ddot{M}}^K(Q) \to N_{\ddot{M}'}^K(Q)$ such that

$$\big(N_f^K(Q)\big)\left(\bigoplus_{\varphi\in K}\mathrm{Ker}\big(\mathrm{Br}^{\ddot{M}}_{\Delta_\varphi(Q)}\big)\right) \subset \bigoplus_{\varphi\in K}\mathrm{Ker}\big(\mathrm{Br}^{\ddot{M}'}_{\Delta_\varphi(Q)}\big);$$

thus, f induces a $\mathfrak{D}(N_G^K(Q) \times N_G^K(Q))$-module homomorphism

$$\bar{N}_f^K(Q): \bar{N}_{\ddot{M}}^K(Q) \to \bar{N}_{\ddot{M}'}^K(Q).$$

3.4. If A is a $\mathfrak{D}G$-interior algebra, A naturally admits a $\mathfrak{D}(G \times G)$-module structure, so that the extended Brauer quotient $\bar{N}_A^K(Q)$ makes sense, and then $\bar{N}_A^K(Q)$ becomes a $\mathfrak{D}N_G^K(Q)$-interior algebra with the distributive product defined by the equality [7, 3.2]

$$\mathrm{Br}^A_{\Delta_\varphi(Q)}(a)\,\mathrm{Br}^A_{\Delta_{\varphi'}(Q)}(a') = \mathrm{Br}^A_{\Delta_{\varphi\circ\varphi'}(Q)}(aa')$$

for any $\varphi, \varphi' \in K$, any $a \in N_A^{\varphi}(Q)$ and any $a' \in N_A^{\varphi'}(Q)$, and with the k-algebra homomorphism

$$\mathfrak{D}N_G^K(Q) \to \bar{N}_A^K(Q)$$

mapping $z \in \mathfrak{D}$ onto $\mathrm{Br}^A_{\Delta_{\mathrm{id}_Q}(Q)}(z\cdot 1_A)$ and $x \in N_G^K(Q)$ onto $\mathrm{Br}^A_{\Delta_{\tilde{x}}(Q)}(x\cdot 1_A)$.

3.5. Some general results in [7, Section 3], as Proposition 3.4, can be easily extended to the new context, but Propositions 3.8 and 3.9 deserve a careful analysis.[2] Let P be a p-subgroup of G containing Q and S a $\mathfrak{D}P$-interior algebra which is a matrix algebra over k and admits a P-stable basis; according to Corollary 5.8 in [4], $S(Q)$ is a matrix algebra over k too (possibly zero!); in particular, $S(Q)$ becomes a $\mathfrak{D}N_P^K(Q)$-interior algebra and at most Q has one local point over the *kP-interior algebra S*. On the other hand, the canonical homomorphism $N_P^K(Q) \to K$ and the trivial $\mathfrak{D}$-interior structure induces a $\mathfrak{D}N_P^K(Q)$-interior algebra structure on kK.

Lemma 3.6. *With the notation and the hypothesis above, assume that Q has a local point χ over the kP-interior algebra S and that the kQ-interior algebra S_χ is K-stable. Then, we have a $\mathfrak{D}N_P^K(Q)$-interior algebra isomorphism*

$$\bar{N}_S^K(Q) \cong S(Q) \otimes_k kK$$

compatible with the K-gradings.

Proof. Since $S(Q)$ is a full matrix algebra over k and a unitary subalgebra of $\bar{N}_S^K(Q)$, it follows from Proposition 2.1 in [2] that the product induces a k-algebra isomorphism

$$S(Q) \otimes_k C_{\bar{N}_S^K(Q)}\big(S(Q)\big) \cong \bar{N}_S^K(Q). \tag{3.6.1}$$

[2] Actually, the fact that Q need not be contained in the K-normalizer of Q has been occasionally forgotten in [7] but the reader will easily modify the concerned statements.

For any $x \in N_P^K(Q)$, the element $\mathrm{Br}^S_{\Delta_{\bar{x}}(Q)}(x \cdot 1_S) \in \bar{N}_S^K(Q)$ obviously induces on $S(Q)$ the same automorphism as $x \cdot 1_{S(Q)} \in S(Q)$; thus, the element $x^{-1} \cdot \mathrm{Br}^S_{\Delta_{\bar{x}}(Q)}(x \cdot 1_S)$ belongs to $C_{\bar{N}_S^K(Q)}(S(Q))$ and it is easily checked that the correspondence mapping $x \in N_P^K(Q)$ onto $x^{-1} \cdot \mathrm{Br}^S_{\Delta_{\bar{x}}(Q)}(x \cdot 1_S)$ defines a group homomorphism

$$N_P^K(Q) \to C_{\bar{N}_S^K(Q)}\big(S(Q)\big)^*.$$

Then, endowed with this group homomorphism and with the trivial $\mathfrak{O}$-interior algebra structure, $C_{\bar{N}_S^K(Q)}(S(Q))$ becomes a $\mathfrak{O}N_P^K(Q)$-interior algebra and isomorphism (3.6.1) a $\mathfrak{O}N_P^K(Q)$-interior algebra isomorphism.

Now, in order to complete the proof, it suffices to show that there exists a $kN_P^K(Q)$-interior algebra isomorphism

$$kK \cong C_{\bar{N}_S^K(Q)}\big(S(Q)\big) \tag{3.6.2}$$

compatible with the K-gradings; actually, since $N_P^K(Q)$ is a p-group and all the terms in the gradings have dimension one, it suffices to exhibit a simple k-algebra isomorphism. Choose $\ell \in \chi$ and set $S_\chi = \ell S \ell$; it follows from Proposition 3.4 in [7] that we have a k-algebra embedding

$$\bar{f} : \bar{N}_{S_\chi}^K(Q) \to \bar{N}_S^K(Q);$$

since $\bar{f}(\ell)$ is a primitive idempotent of $S(Q)$, it follows from isomorphism (3.6.1) that $\bar{f}$ induces a k-algebra isomorphism

$$\bar{N}_{S_\chi}^K(Q) \cong C_{\bar{N}_S^K(Q)}\big(S(Q)\big)$$

compatible with the K-gradings.

But, to assume that S_χ is K-stable amounts to say that, for any $\varphi \in K$, there is $a_\varphi \in (S_\chi)^*$ such that $a_\varphi \cdot u \cdot a_\varphi^{-1} = \varphi(u)$ for any $u \in Q$ and therefore a_φ belongs to $N_{S_\chi}^\varphi(Q)$; thus, since $S_\chi(Q) \cong k$, we get $\bar{N}_{S_\chi}^\varphi(Q) = k \cdot \bar{a}_\varphi$ and isomorphism (3.6.2) follows from Dade's Theorem in [3]. □

Proposition 3.7. *With the notation and the hypothesis above, assume that Q has a local point χ over the kP-interior algebra S and that the kQ-interior algebra S_χ is K-stable. Then, for any $\mathfrak{O}P$-interior algebra B we have a $\mathfrak{O}N_P^K(Q)$-interior algebra isomorphism*

$$\bar{N}_{S \otimes_k B}^K(Q) \cong S(Q) \otimes_k \bar{N}_B^K(Q)$$

compatible with the K-gradings.

Proof. For any $\varphi \in K$, since there is $a_\varphi \in (S_\chi)^*$ such that $a_\varphi \cdot u \cdot a_\varphi^{-1} = \varphi(u)$ for any $u \in Q$, S_χ still admits a $\Delta_\varphi(Q)$-stable basis; moreover, according to [5, 3.13], there is a permutation kQ-module V such that we have a kQ-interior algebra embedding $\mathrm{Res}^P_Q(S) \to \mathrm{End}_k(V) \otimes_k S_\chi$ and therefore S admits a $\Delta_\varphi(Q)$-stable basis too. Hence, we have an isomorphism

$$\bar{N}_{S \otimes_k B}^K(Q) \cong \bigoplus_{\varphi \in K} S\big(\Delta_\varphi(Q)\big) \otimes_k B\big(\Delta_\varphi(Q)\big).$$

But, it is clear that the direct sum

$$\bigoplus_{\varphi\in K} S\bigl(\Delta_\varphi(Q)\bigr)\otimes_k B\bigl(\Delta_\varphi(Q)\bigr)=\bigoplus_{\varphi\in K}\bar N_S^\varphi(Q)\otimes_k \bar N_B^\varphi(Q)$$

is a $\mathfrak{D}N_P^K(Q)$-interior subalgebra of $\bar N_S^K(Q)\otimes_k \bar N_B^K(Q)$; moreover, it follows from Lemma 3.6 above that we have

$$\bar N_S^\varphi(Q)=S(Q)\bar s_\varphi=\bar s_\varphi S(Q)$$

for suitable elements $s_\varphi\in N_S^\varphi(Q)$ fulfilling $\bar s_\varphi\bar s_{\varphi'}=\bar s_{\varphi\circ\varphi'}$ and $s_{\bar x}=x\cdot 1_S$ for any $x\in N_P^K(Q)$. Now, it is easy to build the announced isomorphism. □

3.8. As in [7], the K-extended Brauer quotient $\bar N_A^K(Q)$ will replace the ordinary Brauer quotient $A(Q)$ and, coherently, we will extend to $\bar N_A^K(Q)$ Theorem 13.9 in [6]; thus, let $\rho: G\to\bar G$ be a surjective group homomorphism of kernel W and $\bar Q$ a p-subgroup of $\bar G$, and assume that A is projective as $k(W\times W)$-module and that a Sylow p-subgroup of $G\times_{\bar G} G$ stabilizes a basis of A. It is clear that $A^W(\bar Q)$ has a $\mathfrak{D}C_G(\rho^{-1}(\bar Q))$-interior algebra structure; similarly, denoting by $\mathfrak{C}$ the (possibly empty!) set of all the complements of W in $\rho^{-1}(\bar Q)$, W acts on $\mathfrak{C}$ and $(\prod_{Q\in\mathfrak{C}}A(Q))^W$ also becomes a $\mathfrak{D}C_G(\rho^{-1}(\bar Q))$-interior algebra. Then, Lemma 13.6 in [6] states that the inclusion $A^W\subset A$ induces a $\mathfrak{D}C_G(\rho^{-1}(\bar Q))$-interior algebra isomorphism

$$A^W(\bar Q)\cong\Bigl(\prod_{Q\in\mathfrak{C}}A(Q)\Bigr)^W. \tag{3.8.1}$$

Thus, if $A^W(\bar Q)\neq\{0\}$ then $\mathfrak{C}\neq\emptyset$; in this case, for any complement Q of W in $\rho^{-1}(\bar Q)$, we denote by

$$c_{\rho,Q}(A): A(Q)^{C_W(Q)}\to A^W(\bar Q) \tag{3.8.2}$$

the $\mathfrak{D}C_G(\rho^{-1}(\bar Q))$-interior algebra embedding induced by isomorphism (3.8.1).

Proposition 3.9. *With the notation and the hypothesis above, let $\bar K$ be a subgroup of* $\mathrm{Aut}(\bar Q)$ *and Q a complement of W in $\rho^{-1}(\bar Q)$, and denote by K the image of $\bar K$ in* $\mathrm{Aut}(Q)$ *and by $\rho_Q^K: N_G^K(Q)\to N_{\bar G}^{\bar K}(\bar Q)$ the group homomorphism both determined by ρ. Then there exists a $\mathfrak{D}\rho(N_G^K(Q))$-interior algebra embedding*

$$e_{\rho,Q}^K(A):\mathrm{Ind}_{\rho_Q^K}\bigl(\bar N_A^K(Q)\bigr)\to\bar N^{\bar K}_{\mathrm{Ind}_\rho(A)}(\bar Q)$$

which is compatible with the K- and $\bar K$-gradings and makes commutative the following diagram

$$\begin{CD}
\mathrm{Ind}_{\rho_Q^K}(\bar N_A^K(Q)) @>{e_{\rho,Q}^K(A)}>> \bar N^{\bar K}_{\mathrm{Ind}_\rho(A)}(\bar Q)\\
@A{d_{\rho_Q^K}(A(Q))}AA @AA{(d_\rho(A))(\bar Q)}A\\
A(Q)^{C_W(Q)} @>{c_{\rho,Q}(A)}>> A^W(\bar Q)
\end{CD} \tag{3.9.1}$$

Proof.[3] Since a Sylow p-subgroup of $G \times_{\bar{G}} G$ stabilizes a basis of A and, considered as $k(W \times W)$-module, A is projective, it is easy to check that $A(Q)$ is also projective as $k(C_W(Q) \times C_W(Q))$-module and therefore $A^{W\cdot Q}$ maps onto $A(Q)^{C_W(Q)}$; in particular, the following commutative diagram determines $c_{\rho,Q}(A)$

$$\begin{tikzcd} A(Q)^{C_W(Q)} \arrow[rr, "c_{\rho,Q}(A)"] & & A^W(\bar{Q}) \\ & A^{W\cdot Q} \arrow[ul] \arrow[ur] & \end{tikzcd}$$

More generally, denoting by $\bar{N}$ the image of $Q \cdot N_G^K(Q)$ in $\mathrm{Aut}(W)$, it is clear that $N_A^{\bar{N}}(W)$ can be identified to a $\mathfrak{D}(Q \cdot N_G^K(Q))$-interior subalgebra of A (cf. 2.6) and that $N^K_{N_A^{\bar{N}}(W)}(Q)$ contains $A^{W\cdot Q}$; moreover, denoting by $\bar{\bar{N}}$ the image of $\bar{N}$ in $\mathrm{Aut}(C_W(Q))$ we claim that

$$N^K_{N_A^{\bar{N}}(W)}(Q) \subset N^{\bar{\bar{N}}}_{N_A^K(Q)}\big(C_W(Q)\big);$$

indeed, for any $\varphi \in K$, it is quite clear that any element in $N_A^{\bar{N}}(W)^{\Delta_\varphi(Q)}$ still belongs to $N_A^{\bar{\bar{N}}}(C_W(Q))^{\Delta_\varphi(Q)}$ and that

$$N_A^{\bar{\bar{N}}}\big(C_W(Q)\big)^{\Delta_\varphi(Q)} = \bigoplus_{\sigma \in \bar{\bar{N}}} A^{\Delta_\sigma(C_W(Q))\cdot\Delta_\varphi(Q)} \subset N^{\bar{\bar{N}}}_{N_A^K(Q)}\big(C_W(Q)\big).$$

Hence, we have the unitary $\mathfrak{D}(N_G^K(Q))$-interior algebra homomorphism determined by $d_{\rho_Q^K}(\bar{N}_A^K(Q))$ (cf. 2.6)

$$N^K_{N_A^{\bar{N}}(W)}(Q) \subset N^{\bar{\bar{N}}}_{N_A^K(Q)}\big(C_W(Q)\big) \to N^{\bar{\bar{N}}}_{\bar{N}_A^K(Q)}\big(C_W(Q)\big) \to \mathrm{Ind}_{\rho_Q^K}\big(\bar{N}_A^K(Q)\big).$$

Similarly, $d_\rho(A)$ induces a unitary $\mathfrak{D}(Q \cdot N_G^K(Q))$-interior algebra homomorphism

$$N_A^{\bar{N}}(W) \to \mathrm{Ind}_\rho(A)$$

and therefore we still get a unitary $\mathfrak{D}(N_G^K(Q))$-interior algebra homomorphism

$$N^K_{N_A^{\bar{N}}(W)}(Q) \to \bar{N}^K_{N_A^{\bar{N}}(W)}(Q) \to \bar{N}^{\bar{K}}_{\mathrm{Ind}_\rho(A)}(\bar{Q}).$$

On the other hand, since a Sylow p-subgroup of $W \cdot Q$ stabilizes a basis of A which is projective as kW-module, there is an idempotent i_Q of $A_Q^{W\cdot Q}$ lifting the image $(c_{\rho,Q}(A))(1)$ of the unity element of $A(Q)$ (cf. (3.8.2)); then, setting $B = i_Q A i_Q$, B remains a $k(W \cdot Q)$-interior algebra, and the embedding determined by the inclusion $B \subset A$ induces a $k(C_W(Q) \cdot Q)$-interior algebra isomorphism $\bar{N}_B^K(Q) \cong \bar{N}_A^K(Q)$ and two kQ-interior algebra embeddings

[3] It is an extension of the proof of Theorem 13.9 in [6] from the Brauer quotient to the extended Brauer quotient, except that in [6] the proof of the compatibility with the centralizer interior structures is missed.

$$N_B^{\bar N}(W) \to N_A^{\bar N}(W) \quad \text{and} \quad \mathrm{Ind}_\rho(B) \to \mathrm{Ind}_\rho(A)$$

where we still denote by ρ its restriction to $W \cdot Q$.

At this point, since the following diagram

$$\begin{tikzcd} A(Q)^{C_W(Q)} \cong B(Q)^{C_W(Q)} \cong B^W(\bar Q) \to A^W(\bar Q) \\ B^{W\cdot Q} \arrow[u] \arrow[u] \end{tikzcd}$$

is commutative and also determines $c_{\rho,Q}(A)$, in order to prove the existence of $e^K_{\rho,Q}(A)$, it suffices to prove the existence of a *k-algebra* isomorphism which makes commutative the following diagram

$$\begin{tikzcd} \mathrm{Ind}_{\rho_Q^K}(\bar N_A^K(Q)) \cong \mathrm{Ind}_{\rho_Q^K}(\bar N_B^K(Q)) \cong \bar N^{\bar K}_{\mathrm{Ind}_\rho(B)}(\bar Q) \to \bar N^{\bar K}_{\mathrm{Ind}_\rho(A)}(\bar Q) \\ N^K_{N_B^{\bar N}(W)}(Q) \arrow[u] \arrow[u] \end{tikzcd}$$

where we still denote by ρ_Q^K its restriction to $C_W(Q) \cdot Z(Q)$.

Indeed, the $\mathfrak{D}$-interior structure of both members of the guessed isomorphism is completely determined by the $\mathfrak{D}$-interior structure of $N^K_{N_B^{\bar N}(W)}(Q)$ since the arrows are $\mathfrak{D}$-linear; moreover, although $N^K_{N_B^{\bar N}(W)}(Q)$ need not be a $kN_G^K(Q)$-interior subalgebra of $N^K_{N_A^{\bar N}(W)}(Q)$, for any $x \in N_G^K(Q)$, the arrows send the element $i_Q \cdot x \cdot i_Q$ of $N^K_{N_B^{\bar N}(W)}(Q)$ to

$$1 \otimes \mathrm{Br}_Q(i_Q \cdot x \cdot i_Q) = x \cdot \big(1 \otimes \mathrm{Br}_Q(1)\big) \quad \text{and} \quad \mathrm{Br}_{\bar Q}(1 \otimes i_Q \cdot x \cdot i_Q) = x \cdot \mathrm{Br}_{\bar Q}(1 \otimes i_Q)$$

in $\mathrm{Ind}_{\rho_Q^K}(\bar N_A^K(Q))$ and $\bar N^{\bar K}_{\mathrm{Ind}_\rho(A)}(\bar Q)$ respectively, since $\mathrm{Br}_Q(i_Q) = 1$ and

$$\mathrm{Br}_{\bar Q}(1 \otimes i_Q) = \big(\big(d_\rho(A)\big)(\bar Q) \circ c_{\rho,Q}(A)\big)(1)$$

which is fixed by the action of $N_G^K(Q)$.

Since $i_Q = 1_B$ belongs to $B_Q^{W\cdot Q}$, it follows from Theorem 2.4 that we have a *Higman exoembedding* of $k(W \cdot Q)$-interior algebras

$$\tilde h : B \to \mathrm{Ind}_Q^{W\cdot Q}\big(\mathrm{Res}_Q^{W\cdot Q}(B)\big) = C$$

which clearly induces respective $k(C_W(Q))$- and kQ-interior algebra embeddings

$$\bar N_B^K(Q) \to \bar N_C^K(Q), \qquad \mathrm{Ind}_\rho(B) \to \mathrm{Ind}_\rho(C) \quad \text{and} \quad N_B^{\bar N}(W) \to N_C^{\bar N}(W)$$

and then we get the following commutative diagram

$$\begin{tikzcd}
\mathrm{Ind}_{\rho_Q^K}(\bar N_B^K(Q)) \arrow[r] & \mathrm{Ind}_{\rho_Q^K}(\bar N_C^K(Q)) \\
N^K_{N_B^{\bar N}(W)}(Q) \arrow[r] \arrow[u] \arrow[d] & N^K_{N_C^{\bar N}(W)}(Q) \arrow[u] \arrow[d] \\
\bar N^{\bar K}_{\mathrm{Ind}_\rho(B)}(\bar Q) \arrow[r] & \bar N^{\bar K}_{\mathrm{Ind}_\rho(C)}(\bar Q)
\end{tikzcd}$$

Consequently, since the vertical arrows are unitary, it suffices to show the existence of a *k-algebra* isomorphism which makes commutative the following diagram

$$\begin{tikzcd}
\mathrm{Ind}_{\rho_Q^K}(\bar N_C^K(Q)) \arrow[r, phantom, "\cong"] & \bar N^{\bar K}_{\mathrm{Ind}_\rho(C)}(\bar Q) \\
N^K_{N_C^{\bar N}(W)}(Q) \arrow[u] \arrow[ur] &
\end{tikzcd} \tag{3.9.2}$$

But, on the one hand, it follows from Corollary 12.7 in [6] that

$$\mathrm{Ind}_\rho(C) = \mathrm{Ind}_\rho\big(\mathrm{Ind}_Q^{W\cdot Q}\big(\mathrm{Res}_Q^{W\cdot Q}(B)\big)\big) \cong \mathrm{Res}_Q^{W\cdot Q}(B). \tag{3.9.3}$$

On the other hand, since we clearly have the orthogonal decomposition

$$1_{\bar N_C^K(Q)} = \sum_{x\in C_W(Q)} \mathrm{Br}_Q^C(1\otimes 1\otimes 1)^x \tag{3.9.4}$$

by [6, 2.6.4], the embedding $\bar N_B^K(Q) \to \bar N_C^K(Q)$ induces an isomorphism of $k(C_W(Q)\cdot Z(Q))$-interior algebras

$$\mathrm{Ind}_{Z(Q)}^{C_W(Q)Z(Q)}\big(\bar N_B^K(Q)\big) \cong \bar N_C^K(Q)$$

and, applying again Corollary 12.7 in [6], we get

$$\mathrm{Ind}_{\rho_Q^K}\big(\bar N_C^K(Q)\big) \cong \mathrm{Ind}_{\rho_Q^K}\big(\mathrm{Ind}_{Z(Q)}^{C_W(Q)Z(Q)}\big(\bar N_B^K(Q)\big)\big) \cong \bar N_B^K(Q). \tag{3.9.5}$$

Finally, it suffices to prove that the images in $\bar N_B^K(Q)$ of an element of $N^K_{N_C^{\bar N}(W)}(Q)$ through the arrows of diagram (3.9.2) and the isomorphisms (3.9.3) and (3.9.5) coincide. But, it is quite clear that an element a of $N_C^{\bar N}(W)$ has the form

$$a = \sum_{\sigma\in\bar N}\sum_{w\in W} \mathrm{Tr}_1^{\Delta_\sigma(W)}(w\otimes a_{\sigma,w}\otimes 1)$$

and its image in $\mathrm{Ind}_\rho(C)$ is equal to

$$1 \otimes a = 1 \otimes \mathrm{Tr}_1^W\left(1 \otimes \sum_{\sigma \in \bar{N}} \sum_{w \in W} a_{\sigma,w} \otimes 1\right);$$

similarly, if $\varphi \in K$ and a is fixed by $\Delta_\varphi(Q)$, since $\Delta_\varphi(Q)$ stabilizes the following decomposition

$$N_C^{\bar{N}}(W) = \bigoplus_{\sigma \in \bar{N}} \bigoplus_{w \in W} \mathrm{Tr}_1^{\Delta_\sigma(W)}(w \otimes B \otimes 1),$$

the image of $a \in N_C^{\bar{N}}(W)^{\Delta_\varphi(Q)}$ in $\bar{N}^K_{N_C^{\bar{N}}(W)}(Q)$ is equal to

$$\mathrm{Br}_{\Delta_\varphi(Q)}^{N_C^{\bar{N}}(W)}(a) = \mathrm{Br}_{\Delta_\varphi(Q)}^{N_C^{\bar{N}}(W)}\left(\sum_{\sigma \in \bar{N}} \sum_{w \in C_W(Q)} \mathrm{Tr}_1^{\Delta_\sigma(W)}(w \otimes a_{\sigma,w} \otimes 1)\right)$$

and therefore its image in $\bar{N}^{\bar{K}}_{\mathrm{Ind}_\rho(C)}(\bar{Q})$ coincides with

$$\mathrm{Br}_{\Delta_\varphi(Q)}^{\mathrm{Ind}_\rho(C)}(1 \otimes a) = \mathrm{Br}_{\Delta_\varphi(Q)}^{\mathrm{Ind}_\rho(C)}\left(1 \otimes \mathrm{Tr}_1^W\left(1 \otimes \sum_{\sigma \in \bar{N}} \sum_{w \in C_W(Q)} a_{\sigma,w} \otimes 1\right)\right)$$

which, according to Corollary 12.7 in [6], the isomorphism (3.9.3) maps onto the element $\sum_{\sigma \in \bar{N}} \sum_{w \in C_W(Q)} a_{\sigma,w}$ of $\bar{N}_B^K(Q)$.

On the other hand, since any $\sigma \in \bar{N}$ stabilizes $C_W(Q)$, it follows from decomposition (3.9.4) that the image of $a \in N_C^{\bar{N}}(W)^{\Delta_\varphi(Q)}$ in $\bar{N}_C^K(Q)$ is equal to

$$\mathrm{Br}_{\Delta_\varphi(Q)}^C(a) = \mathrm{Br}_{\Delta_\varphi(Q)}^C\left(\sum_{\sigma \in \bar{N}} \sum_{w, w' \in C_W(Q)} \sigma(w')w \otimes a_{\sigma,w} \otimes w'^{-1}\right)$$

and therefore its image in $\mathrm{Ind}_{\rho_Q^K}(\bar{N}_C^K(Q))$ is equal to

$$1 \otimes \mathrm{Br}_{\Delta_\varphi(Q)}^C(a) = 1 \otimes \mathrm{Tr}_1^{C_W(Q)}\left(1 \otimes \sum_{\sigma \in \bar{N}} \sum_{w \in C_W(Q)} a_{\sigma,w} \otimes 1\right)$$

which, according to Corollary 12.7 in [6], the isomorphism (3.9.5) maps onto the element $\sum_{\sigma \in \bar{N}} \sum_{w \in C_W(Q)} a_{\sigma,w}$ of $\bar{N}_B^K(Q)$. We are done. □

Remark 3.10. In the proof above, it is possible to choose the idempotent i_Q, which lifts $(c_{\rho,Q}(A))(1)$ to $A_Q^{W\cdot Q}$, fixed by $N_G^K(Q)$ since the natural homomorphism $A^{W\cdot Q} \to (A^W)(Q)$ is actually a so-called *covering homomorphism of $N_G(Q)$-algebras* [4, Section 4] and then it suffices to apply Proposition 4.18 in [4].

In order to prove our claim, we choose a Sylow p-subgroup S of G containing a Sylow p-subgroup of $N_G(Q)$, and then, setting $T = W \cap S$, we choose a $(T \times T) \cdot \Delta(S)$-stable basis X of A. It is easily checked that the subset $W \otimes X \otimes W$ of $B = \mathrm{Ind}_S^{W\cdot S}(A)$ is a $(W \times W) \cdot \Delta(S)$-stable basis of B; from this basis, it is not difficult to compute bases of $B^{W\cdot R}$ and of $(B^W)(Q)^R$ for any subgroup of $N_S(Q)$ containing Q, showing that $B^{W\cdot R}$ maps onto $(B^W)(Q)^R$. Finally, since p does not divide $|(W \cdot S) : S|$, the canonical $k(W \cdot S \times W \cdot S)$-module injection $A \to B$ is split, so that $A^{W\cdot R}$ still maps onto $(A^W)(Q)^R$. We are done.

Lemma 3.11. *If a $\mathfrak{D}G$-interior algebra A is a 0-split $\mathfrak{D}(G \times G)$-module then, for any p-subgroup Q of G and any subgroup K of* $\mathrm{Aut}(Q)$, *the $\mathfrak{D}N_G^K(Q)$-interior algebra $\bar{N}_A^K(Q)$ is a 0-split $\mathfrak{D}(N_G^K(Q) \times N_G^K(Q))$-module and the inclusion $N_{\mathbb{C}_0(A)}^K(Q) \subset \mathbb{C}_0(N_A^K(Q))$ induces a $kN_G^K(P)$-interior algebra isomorphism*

$$\bar{N}_{\mathbb{H}_0(A)}^K(Q) \cong \mathbb{H}_0\big(\bar{N}_A^K(Q)\big).$$

Proof. By the very definition of a 0-split $\mathfrak{D}(G \times G)$-module (cf. 1.7), there are a $k(G \times G)$-module M and a contractible $\mathfrak{D}(G \times G)$-module C such that we have a $\mathfrak{D}(G \times G)$-module isomorphism

$$A \cong M \oplus C \tag{3.11.1}$$

where we consider the trivial $\mathfrak{D}$-interior structure on M; in particular, we have $k(G \times G)$-module isomorphism

$$\mathbb{C}_0(A) \cong M \oplus \mathbb{C}_0(C) \quad \text{and} \quad \mathbb{B}_0(A) \cong \mathbb{B}_0(C) = \mathbb{C}_0(C)$$

and therefore $\mathbb{H}_0(A) \cong M$ as $\mathfrak{D}(G \times G)$-modules.

For any p-subgroup Q and any subgroup K of $\mathrm{Aut}(Q)$, isomorphism (3.11.1) induces a $\mathfrak{D}(N_G^K(P) \times N_G^K(P))$-module isomorphism

$$\bar{N}_A^K(Q) \cong \bar{N}_M^K(Q) \oplus \bar{N}_C^K(Q) \tag{3.11.2}$$

and we claim that $\bar{N}_C^K(Q)$ is a contractible $\mathfrak{D}(N_G^K(Q) \times N_G^K(Q))$-module. Indeed, by the very definition of contractibility, there exists $h \in \mathrm{End}_{k(G\times G)}(C)$ such that $\mathrm{id}_C = d \cdot h + h \cdot d$ where $d \in \mathfrak{D}$; but, for any $\varphi \in K$, it is clear that $h(C^{\Delta_\varphi(Q)}) \subset C^{\Delta_\varphi(Q)}$ and therefore h induces a k-endomorphism h_φ of $C^{\Delta_\varphi(Q)}$ which also fulfills $\mathrm{id}_{C^{\Delta_\varphi(Q)}} = d \cdot h_\varphi + h_\varphi \cdot d$; moreover, it is easily checked that $\bigoplus_{\varphi\in K} h_\varphi$ is a $\mathfrak{D}(N_G^K(Q) \times N_G^K(Q))$-module endomorphism of $N_C^K(Q)$; hence, $N_C^K(Q)$ is a contractible $\mathfrak{D}(N_G^K(Q) \times N_G^K(Q))$-module. Pushing it further, since h is compatible with the relative trace maps, $\bar{N}_C^K(Q)$ is a contractible $\mathfrak{D}(N_G^K(Q) \times N_G^K(Q))$-module too.

Consequently, it follows from isomorphism (3.11.2) that $\bar{N}_A^K(Q)$ is a 0-split $\mathfrak{D}(N_G^K(Q) \times N_G^K(Q))$-module and, applying the argument above, we have a $k(N_G^K(Q) \times N_G^K(Q))$-module isomorphism $\bar{N}_{\mathbb{H}_0(A)}^K(Q) \cong \mathbb{H}_0(\bar{N}_A^K(Q))$; moreover, it is easily checked that this isomorphism makes commutative the following diagram

$$\begin{array}{ccc} \bar{N}_{\mathbb{H}_0(A)}^K(Q) & \cong & \mathbb{H}_0(\bar{N}_A^K(Q)) \\ \uparrow & & \uparrow \\ N_{\mathbb{C}_0(A)}^K(Q) & \longrightarrow & \mathbb{C}_0(N_A^K(Q)) \end{array}$$

where the vertical arrows are surjective. □

4. Proof of Theorem 1.12

4.1. Let us come back to our standard setting (cf. 1.8) and assume that kGb and $kG'b'$ are *basic Rickard equivalent*. Let (P, e) be a maximal Brauer (b, G)-pair and (Q, f) a Brauer (b, G)-pair contained in (P, e); note that, for any subgroup H of $N_G(Q, f)$ containing $C_G(Q)$, f is also a block of H.

4.2. As in Theorem 1.12, let K be a subgroup of the image of $N_G(Q,f)$ in $\mathrm{Aut}(Q)$, so that f is also a block of both $N_G^K(Q)$ and $Q\cdot N_G^K(Q)$. Let (R,g) and (T,h) be maximal Brauer $(f,Q\cdot N_G^K(Q))$- and $(f,Q\cdot C_G(Q))$-pairs respectively; since R and T contain Q, (R,g) and (T,h) are also Brauer (b,G)-pairs and, without loss of generality, we may assume that

$$(Q,f)\subset(T,h)\subset(R,g)\subset(P,e),$$

so that we clearly have $R=Q\cdot N_P^K(Q)$ and $T=Q\cdot C_P(Q)$; actually, it is easily checked that both (R,g) and (T,h) are *selfcentralizing* (cf. 1.10). Let us denote by γ, ε and ν the local points of P, R and T over kGb associated with e, g and h respectively, so that we have $T_\nu\subset R_\varepsilon\subset P_\gamma$; borrowing our notation from 2.7 above, since we assume that kGb and $kG'b'$ are *Rickard equivalent*, it follows from Proposition 2.8 that γ, ε and ν respectively determine *noncontractile* local points $\hat\gamma$, $\hat\varepsilon$ and $\hat\nu$ of P, R and T over $\hat A$ still fulfilling $T_{\hat\nu}\subset R_{\hat\varepsilon}\subset P_{\hat\gamma}$.

4.3. Then, it follows from Theorem 16.15 in [6] that we have a local tracing triple $(P_{\hat\gamma},\ddot P_{\ddot\gamma},P'_{\gamma'})$; moreover, since we are assuming that the above equivalence is *basic*, each one of the groups $\ddot P\times_P\ddot P$ and $\ddot P\times_{P'}\ddot P$ stabilizes a basis of $\mathrm{End}_k(\ddot N)$ and thus, according to our choice, each of them stabilizes a basis of $\ddot S$ too (cf. 2.7); then, by [6, 19.3], we know that $(P_{\hat\gamma},\ddot P_{\ddot\gamma},P'_{\gamma'})$ is *basic* (cf. 2.11). In particular, the group homomorphism $\sigma:\ddot P\to P$ determined by the first projection $G\times G'\to G$ admits a section and we choose a *section* $\lambda:P\to\ddot P$ of $(P_{\hat\gamma},\ddot P_{\ddot\gamma},P'_{\gamma'})$ in the sense of 2.11 above. Moreover, denoting by $\sigma':\ddot P\to P'$ the group homomorphism determined by the second projection $G\times G'\to G'$ and setting $P^\lambda=\lambda(\ddot P)$, $\lambda'=\sigma'\circ\lambda$ and $P^{\lambda'}=\lambda'(P)$, the group homomorphism $\lambda':P\to P'$ is injective since $\ddot A_{\ddot\gamma}(P^\lambda)\neq 0$ [6, 17.4.7]. Thus, by symmetry, λ' is an isomorphism and we have $P^{\lambda'}=P'$,

4.4. At this point, it follows from Theorem 2.14 above, applied to the local pointed group $R_{\hat\varepsilon}$ on $\hat A$ contained in $P_{\hat\gamma}$, that we have a basic local tracing triple $(R_{\hat\varepsilon},\ddot R_{\ddot\varepsilon},R'_{\varepsilon'})$ contained in $(P_{\hat\gamma},\ddot P_{\ddot\gamma},P'_{\gamma'})$ and admitting the restriction of λ to R as a section; the same argument applied to the local pointed group $T_{\hat\nu}$ contained in $R_{\hat\varepsilon}$ and to the local tracing triple $(R_{\hat\varepsilon},\ddot R_{\ddot\varepsilon},R'_{\varepsilon'})$ provides a basic local tracing triple $(T_{\hat\nu},\ddot T_{\ddot\nu},T'_{\nu'})$ on $\hat A$, $\ddot A$ and kG', such that finally we get the inclusions

$$\bigl(T_{\hat\nu},\ddot T_{\ddot\nu},T'_{\nu'}\bigr)\subset\bigl(R_{\hat\varepsilon},\ddot R_{\ddot\varepsilon},R'_{\varepsilon'}\bigr)\subset\bigl(P_{\hat\gamma},\ddot P_{\ddot\gamma},P'_{\gamma'}\bigr)$$

and we know that the restrictions of λ to R and T are respective sections of the triples $(R_{\hat\varepsilon},\ddot R_{\ddot\varepsilon},R'_{\varepsilon'})$ and $(T_{\hat\nu},\ddot T_{\ddot\nu},T'_{\nu'})$. Moreover, according to Theorem 18.8 in [6], $P'_{\gamma'}$ is a defect pointed group of b' and, denoting by e' and g' the respective blocks of $C_{G'}(P')$ and $C_{G'}(R')$ determined by γ' and ε', we have

$$(R',g')\subset(P',e').$$

4.5. We set $R^{\lambda'}=\lambda'(R)\subset R'$, $T^{\lambda'}=\lambda'(T)\subset T'$ and $Q^{\lambda'}=\lambda'(Q)\subset Q'$ (which need not agree with the notation of Theorem 1.12!), and respectively denote by $g^{\lambda'}$, $h^{\lambda'}$ and $f^{\lambda'}$ the blocks of $C_{G'}(R^{\lambda'})$, $C_{G'}(T^{\lambda'})$ and $C_{G'}(Q^{\lambda'})$ fulfilling [6, 2.13.2]

$$\bigl(Q^{\lambda'},f^{\lambda'}\bigr)\subset\bigl(T^{\lambda'},h^{\lambda'}\bigr)\subset\bigl(R^{\lambda'},g^{\lambda'}\bigr)\subset(R',g')\subset(P',e');$$

in particular, since R_ε is selfcentralizing (cf. 4.2), it follows from Corollary 19.9 in [6] that $R^{\lambda'}$ has a unique local point $\varepsilon^{\lambda'}$ over kG' such that $R^{\lambda'}_{\varepsilon^{\lambda'}}\subset R'_{\varepsilon'}$ and that $R^{\lambda'}_{\varepsilon^{\lambda'}}$ and $R'_{\varepsilon'}$ are both selfcentralizing too; then, it follows from Lemma 3.9 in [4], from Theorem 19.7 in [6] and from Lemma 4.18 below that $R^{\lambda'}$ is a defect group of $f^{\lambda'}$ as a block of $Q^{\lambda'}\cdot N_{G'}^{K^{\lambda'}}(Q^{\lambda'})$ and therefore that $N_{P'}^{K^{\lambda'}}(Q^{\lambda'})$ is a defect group of $f^{\lambda'}$ as a block of $N_{G'}^{K^{\lambda'}}(Q^{\lambda'})$.

4.6. Denote by $\rho : \ddot{R} \to R$ and $\rho' : \ddot{R} \to R'$ the respective group homomorphisms determined by the first and the second projections $G \times G' \to G$ and $G \times G' \to G'$; by the very definition of *local tracing triples* (cf. 2.9), there exists a unique $\mathfrak{D}R$-interior algebra exoembedding

$$\tilde{h}_{\hat{\varepsilon}}^{\ddot{\varepsilon},\varepsilon'} : \hat{A}_{\hat{\varepsilon}} \to \operatorname{Ind}_\rho\big(\ddot{A}_{\ddot{\varepsilon}} \otimes_k \operatorname{Res}_{\rho'}\big((kG')_{\varepsilon'}\big)\big)$$

such that we have the corresponding commutative diagram in 2.9; then, setting

$$B_{\ddot{\varepsilon},\varepsilon'} = \ddot{A}_{\ddot{\varepsilon}} \otimes_k \operatorname{Res}_{\rho'}\big((kG')_{\varepsilon'}\big),$$

by Proposition 3.4 in [7] suitably extended to $\mathfrak{D}$-interior algebras, $h_{\hat{\varepsilon}}^{\ddot{\varepsilon},\varepsilon'}$ induces a $\mathfrak{D}N_P^K(Q)$-interior algebra exoembedding which extends $\tilde{h}_{\hat{\varepsilon}}^{\ddot{\varepsilon},\varepsilon'}(Q)$

$$\bar{N}^K_{\tilde{h}_{\hat{\varepsilon}}^{\ddot{\varepsilon},\varepsilon'}}(Q) : \bar{N}^K_{\hat{A}_{\hat{\varepsilon}}}(Q) \to \bar{N}^K_{\operatorname{Ind}_\rho(B_{\ddot{\varepsilon},\varepsilon'})}(Q).$$

4.7. On the other hand, denoting by K^λ the image of K in $\operatorname{Aut}(Q^\lambda)$ determined by λ, Proposition 3.9 applied to $\pi : G \times G' \to G$ supplies a $\mathfrak{D}N_P^K(Q)$-interior algebra embedding

$$e^{K^\lambda}_{\rho,Q^\lambda}(B_{\ddot{\varepsilon},\varepsilon'}) : \operatorname{Ind}_{\rho^{K^\lambda}_{Q^\lambda}}\big(\bar{N}^{K^\lambda}_{B_{\ddot{\varepsilon},\varepsilon'}}\big(Q^\lambda\big)\big) \to \bar{N}^K_{\operatorname{Ind}_\rho(B_{\ddot{\varepsilon},\varepsilon'})}(Q).$$

Now, we have two idempotents in $(\operatorname{Ind}_\rho(B_{\ddot{\varepsilon},\varepsilon'})(Q))^{N_P^K(Q)}$, namely the two images $(\bar{N}^K_{h_{\hat{\varepsilon}}^{\ddot{\varepsilon},\varepsilon'}}(Q))(1)$ and $(e^{K^\lambda}_{\rho,Q^\lambda}(B_{\ddot{\varepsilon},\varepsilon'}))(1)$ of the respective unity elements of $\bar{N}^K_{\hat{A}_{\hat{\varepsilon}}}(Q)$ and $\operatorname{Ind}_{\rho^{K^\lambda}_{Q^\lambda}}(\bar{N}^{K^\lambda}_{B_{\ddot{\varepsilon},\varepsilon'}}(Q^\lambda))$; moreover, since $(R_{\hat{\varepsilon}}, \ddot{R}_{\ddot{\varepsilon}}, R'_{\varepsilon'})$ is basic, R stabilizes a basis in $\hat{A}_{\hat{\varepsilon}}$ [6, 17.1] and therefore the first idempotent is primitive. Consequently, $\bar{N}^K_{\tilde{h}_{\hat{\varepsilon}}^{\ddot{\varepsilon},\varepsilon'}}(Q)$ factorizes throughout $\tilde{e}^{K^\lambda}_{\rho,Q^\lambda}(B_{\ddot{\varepsilon},\varepsilon'})$ determining a unique $\mathfrak{D}N_P^K(Q)$-interior algebra exoembedding

$$\tilde{h}^\lambda_{\hat{\varepsilon}}(Q) : \bar{N}^K_{\hat{A}_{\hat{\varepsilon}}}(Q) \to \operatorname{Ind}_{\rho^{K^\lambda}_{Q^\lambda}}\big(\bar{N}^{K^\lambda}_{B_{\ddot{\varepsilon},\varepsilon'}}\big(Q^\lambda\big)\big) \tag{4.7.1}$$

always compatible with the K-grading.

4.8. Once again because $(R_{\hat{\varepsilon}}, \ddot{R}_{\ddot{\varepsilon}}, R'_{\varepsilon'})$ is basic, $\ddot{R}$ stabilizes a basis in $\ddot{A}_{\ddot{\varepsilon}}$ [6, 17.1] and, since $\ddot{A}_{\ddot{\varepsilon}}$ is actually a matrix algebra over k (cf. 2.7), we claim that we can apply Proposition 3.7 to $\ddot{A}_{\ddot{\varepsilon}}$ and $\operatorname{Res}_{\rho'}((kG')_{\varepsilon'})$ obtaining a $\mathfrak{D}N_{\ddot{R}}^{K^\lambda}(Q^\lambda)$-interior algebra isomorphism

$$\bar{N}^{K^\lambda}_{B_{\ddot{\varepsilon},\varepsilon'}}\big(Q^\lambda\big) \cong \ddot{A}_{\ddot{\varepsilon}}\big(Q^\lambda\big) \otimes_k \bar{N}^{K^{\lambda'}}_{\operatorname{Res}_{\rho'}((kG')_{\varepsilon'})}\big(Q^{\lambda'}\big)$$

where $K^{\lambda'}$ denotes the image of K in $\operatorname{Aut}(Q^{\lambda'})$. Indeed, by the so-called *Frattini argument*, we have

$$Q \cdot N_G^K(Q) = Q \cdot C_G(Q) \cdot \big(N_G^K(Q) \cap N_G(T,h)\big)$$

and therefore, since (T,h) is selfcentralizing, any $\varphi \in K$ is induced by some element $x \in N_{N_G^K(Q)}(T_\nu)$; then, it follows from Theorem 16.9 and Corollary 16.16 in [6] that there is $x' \in G'$ such that (x,x') normalizes the triple $(T_{\hat{\nu}}, \ddot{T}_{\ddot{\nu}}, T'_{\nu'})$; thus, denoting by μ the restriction of λ to T, it is clear that

$\mu^{(x,x')}$ is also a section of the triple $(T_{\hat\nu}, \ddot T_{\ddot\nu}, T'_{\nu'})$ and it follows from 2.12.1 that there is $y' \in G'$ fulfilling

$$\ddot T^{(1,y')} = \ddot T \quad \text{and} \quad \mu^{(x,x'y')} = \mu$$

and therefore $(x, x'y')$ normalizes T^λ and Q^λ and induces φ on Q^λ.

4.9. But, since $\ddot T_{\ddot\nu} \subset \ddot R_{\ddot\varepsilon}$, we have a $\mathfrak{D}\ddot T$-embedding $\ddot A_{\ddot\nu} \to \mathrm{Res}^{\ddot R}_{\ddot Q}(\ddot A_{\ddot\varepsilon})$ and therefore, since $\ddot A_{\ddot\nu}(T^\lambda) \neq \{0\}$ [6, 17.4.7] and $\ddot A_{\ddot\varepsilon}$ admits an $\ddot R$-stable basis, we still have $\ddot A_{\ddot\varepsilon}(Q^\lambda) \neq \{0\}$ [6, Lemma 7.10]; on the other hand, if $\ddot X$ is a $\ddot T$-stable basis of $\ddot A_{\ddot\nu}$, it is easily checked that $(1 \times G') \otimes \ddot X \otimes (1 \times G')$ is a $\ddot T$-stable basis of $\mathrm{Ind}_{\ddot T}^{T\times G'}(\ddot A_{\ddot\nu})$ and, in particular, it follows again from Corollary 5.8 in [4] that Q^λ has a unique local point χ^λ over the $k(T \times G')$-*interior algebra* $\mathrm{Ind}_{\ddot T}^{T\times G'}(\ddot A_{\ddot\nu})$, which necessarily comes from $\ddot A_{\ddot\nu}$; moreover, it is clear that $(1, y')$ acts on $\mathrm{Ind}_{\ddot T}^{T\times G'}(\ddot A_{\ddot\nu})$ by conjugation, whereas the action of (x, x') on the group $T \times G'$ normalizing $\ddot T_{\ddot\nu}$ still determines a *twisted* action over this $k(T \times G')$-interior algebra; at this point, the uniqueness of χ implies that the action of $(x, x'y')$ normalizes $(Q^\lambda)_{\chi^\lambda}$; that is to say, we get

$$\mathrm{Res}_\varphi\big((\ddot A_{\ddot\varepsilon})_{\chi^\lambda}\big) = \mathrm{Res}_\varphi\big((\ddot A_{\ddot\nu})_{\chi^\lambda}\big) \cong (\ddot A_{\ddot\nu})_{\chi^\lambda} = (\ddot A_{\ddot\varepsilon})_{\chi^\lambda}$$

which proves our claim.

4.10. Thus, setting $\ddot D = \ddot A_{\ddot\varepsilon}(Q^\lambda)$ and $D' = \bar N^{K^{\lambda'}}_{(kG')_{\varepsilon'}}(Q^{\lambda'})$, the $\mathfrak{D}N_P^K(Q)$-interior algebra exoembedding (4.7.1) becomes

$$\tilde h^\lambda_{\hat\varepsilon}(Q) : \bar N^K_{\hat A_{\hat\varepsilon}}(Q) \to \mathrm{Ind}_{\rho^{K^\lambda}_{Q^\lambda}}\big(\ddot D \otimes_k \mathrm{Res}_{\rho^{K^{\lambda'}}_{Q^{\lambda'}}}(D')\big); \tag{4.10.1}$$

once again, since $(\tilde h^\lambda_{\hat\varepsilon}(Q))(1)$ belongs to a local point of $N_P^K(Q)$, it is clear that, for suitable points $\ddot\beta$ of $N^{K^\lambda}_{\ddot R}(Q^\lambda)$ over $\ddot D$ and β' of $N^{K^{\lambda'}}_{P'}(Q^{\lambda'})$ over D', $\tilde h^\lambda_{\hat\varepsilon}(Q)$ factorizes throughout the canonical exoembedding

$$\mathrm{Ind}_{\rho^{K^\lambda}_{Q^\lambda}}\big(\ddot D_{\ddot\beta} \otimes_k \mathrm{Res}_{\rho^{K^{\lambda'}}_{Q^{\lambda'}}}(D'_{\beta'})\big) \to \mathrm{Ind}_{\rho^{K^\lambda}_{Q^\lambda}}\big(\ddot D \otimes_k \mathrm{Res}_{\rho^{K^{\lambda'}}_{Q^{\lambda'}}}(D')\big)$$

determining a unique $\mathfrak{D}N_P^K(Q)$-interior algebra exoembedding

$$\tilde h^{\ddot\beta,\beta'}_{\hat\varepsilon}(Q) : \bar N^K_{\hat A_{\hat\varepsilon}}(Q) \to \mathrm{Ind}_{\rho^{K^\lambda}_{Q^\lambda}}\big(\ddot D_{\ddot\beta} \otimes_k \mathrm{Res}_{\rho^{K^{\lambda'}}_{Q^{\lambda'}}}(D'_{\beta'})\big). \tag{4.10.2}$$

4.11. We claim that β' is a local point; indeed, let $T'_{\eta'}$ be a defect pointed group of $N^{K^{\lambda'}}_{P'}(Q^{\lambda'})_{\beta'}$; then, it follows from Theorem 2.4 that there exists a $\mathfrak{D}N^{K^{\lambda'}}_{P'}(Q^{\lambda'})$-interior algebra exoembedding

$$\tilde h^{\eta'}_{\beta'} : D'_{\beta'} \to \mathrm{Ind}_{T'}^{N^{K^{\lambda'}}_{P'}(Q^{\lambda'})}\big(D'_{\eta'}\big);$$

note that, since $N^{K^{\lambda'}}_{P'}(Q^{\lambda'})$ is a p-group, $\tilde h^{\eta'}_{\beta'}$ is actually an exoisomorphism [6, 2.12.2]. Consequently, setting $\ddot T = \rho'^{-1}(N^{K^{\lambda'}}_{P'}(Q^{\lambda'}))$ and denoting by $\tau' : \ddot T \to T'$ the homomorphism determined by ρ'

(cf. 4.6), it follows from Proposition 12.12 in [6] that we have

$$\mathrm{Res}_{\rho^{K^{\lambda'}}_{Q^{\lambda'}}}(D'_{\beta'}) \cong \mathrm{Ind}_{\ddot{T}}^{N^{K^\lambda}_{\ddot{R}}(Q^\lambda)}\big(\mathrm{Res}_{\tau'}(D'_{\eta'})\big);$$

then, by Corollary 12.7 and Proposition 12.9 in [6], we get a $\mathfrak{D}N^K_P(Q)$-interior algebra exoembedding

$$\tilde{h}^{\ddot{\eta},\beta'}_{\hat{\varepsilon}}(Q) : \bar{N}^K_{\hat{A}_{\hat{\varepsilon}}}(Q) \to \mathrm{Ind}_\tau\big(\mathrm{Res}_{\ddot{T}}^{N^{K^\lambda}_{\ddot{R}}(Q^\lambda)}(\ddot{D}_{\ddot{\beta}}) \otimes_k \mathrm{Res}_{\tau'}(D'_{\eta'})\big)$$

where we denote by $\tau : \ddot{T} \to N^K_P(Q)$ the group homomorphism determined by ρ (cf. 4.6).

4.12. Since the unity element forms a local point of $N^K_P(Q)$ on $\bar{N}^K_{\hat{A}_{\hat{\varepsilon}}}(Q)$, we necessarily have $\tau(\ddot{T}) = N^K_P(Q)$ and therefore we still have

$$N^{K^\lambda}_{\ddot{R}}(Q^\lambda) = N^{K^\lambda}_{\mathrm{Ker}(\rho)}(Q^\lambda) \cdot \ddot{T};$$

moreover, since $\ddot{T} \subset N^{K^\lambda}_{\ddot{R}}(Q^\lambda)$ stabilizes a basis of $\ddot{D}_{\ddot{\beta}} \otimes_k \mathrm{Res}_{\rho^{K^{\lambda'}}_{Q^{\lambda'}}}(D'_{\beta'})$ and $\ddot{D}_{\ddot{\beta}}$ is projective as $\mathrm{Ker}(\tau)$-module, it follows from Lemma 13.6 in [6] that $\mathrm{Ker}(\tau)$ has at least one complement $\ddot{U}$ in $\ddot{T}$ such that

$$\big(\ddot{D}_{\ddot{\beta}} \otimes_k \mathrm{Res}_{\rho^{K^{\lambda'}}_{Q^{\lambda'}}}(D'_{\beta'})\big)(\ddot{U}) \neq \{0\};$$

thus, since $\ddot{D}_{\ddot{\beta}}$ is also projective as $\mathrm{Ker}(\tau')$-module, we have $\ddot{U} \cap \mathrm{Ker}(\tau') = \{1\}$ and therefore $\ddot{U}$ maps into T', so that $|N^K_P(Q)| = |\ddot{U}| \leqslant |T'|$ which forces the equalities $T' = N^{K^{\lambda'}}_{P'}(Q^{\lambda'})$ and $\eta' = \beta'$.

4.13. Furthermore, since $(Q^{\lambda'}, f^{\lambda'}) \subset (R', g')$, we have [6, 2.13.2]

$$f^{\lambda'}\,\mathrm{Br}_Q(\varepsilon') = \mathrm{Br}_Q(\varepsilon');$$

hence, $N^{K^{\lambda'}}_{P'}(Q^{\lambda'})_{\beta'}$ is a maximal local pointed group over $kN^{K^{\lambda'}}_{G'}(Q^{\lambda'})f^{\lambda'}$ (cf. 4.5), so that $(kN^{K^{\lambda'}}_{G'}(Q^{\lambda'}))_{\beta'}$ is a *source algebra* of $kN^{K^{\lambda'}}_{G'}(Q^{\lambda'})f^{\lambda'}$, and the $\mathfrak{D}N^K_P(Q)$-interior algebra exoembedding (4.10.2) becomes

$$\tilde{h}^{\ddot{\beta},\beta'}_{\hat{\varepsilon}}(Q) : \bar{N}^K_{\hat{A}_{\hat{\varepsilon}}}(Q) \to \mathrm{Ind}_{\rho^{K^\lambda}_{Q^\lambda}}\big(\ddot{D}_{\ddot{\beta}} \otimes_k \mathrm{Res}_{\rho^{K^{\lambda'}}_{Q^{\lambda'}}}\big(kN^{K^{\lambda'}}_{G'}(Q^{\lambda'})\big)_{\beta'}\big). \tag{4.13.1}$$

4.14. On the other hand, it follows from Lemma 4.18 below that there is a local point $\bar{\varepsilon}$ of $N^K_R(Q)$ over $kN^K_G(Q)f$ such that the structural homomorphism from $N^K_G(Q)$ to $N^K_{kG}(Q)$ induces a $kN^K_R(Q)$-interior algebra embedding

$$\big(kN^K_G(Q)\big)_{\bar{\varepsilon}} \to \bar{N}^K_{(kG)_\varepsilon}(Q); \tag{4.14.1}$$

moreover, according to our choice of R_ε (cf. 4.2) and to the same lemma, $N^K_R(Q)_{\bar{\varepsilon}}$ is a maximal local pointed group over $kN^K_G(Q)f$ and therefore $N^K_P(Q)$ is a defect group of f as a block of $N^K_G(Q)$, and the $kN^K_P(Q)$-interior algebra $(kN^K_G(Q))_{\bar{\varepsilon}}$ is a *source algebra* of $kN^K_G(Q)f$.

4.15. Finally, according to our hypothesis (cf. 2.12, 4.1 and 4.3), $\hat{A}_{\hat{\varepsilon}}$ is a 0-split $\mathfrak{D}(P \times P)$-module and it follows from Proposition 2.8 that we have a kR-interior algebra isomorphism

$$(kG)_\varepsilon \cong \mathrm{H}_0(\hat{A}_{\hat{\varepsilon}});$$

then, by Lemma 3.11 applied to the kR-interior algebra $\hat{A}_{\hat{\varepsilon}}$, $\bar{N}^K_{\hat{A}_{\hat{\varepsilon}}}(Q)$ is a 0-split $\mathfrak{D}(N^K_P(Q) \times N^K_P(Q))$-module and we get a $kN^K_P(Q)$-interior algebra embedding (cf. (4.14.1))

$$\big(kN^K_G(Q)\big)_{\bar{\varepsilon}} \to \bar{N}^K_{(kG)_\varepsilon}(Q) \cong \mathrm{H}_0\big(\bar{N}^K_{\hat{A}_{\hat{\varepsilon}}}(Q)\big).$$

4.16. That is to say, denoting by $\hat{\bar{\varepsilon}}$ the local point of $N^K_P(Q)$ over

$$\mathrm{Ind}_{\rho^{K^\lambda}_{Q^\lambda}}\big(\ddot{D}_{\ddot{\beta}} \otimes_k \mathrm{Res}_{\rho^{K^{\lambda'}}_{Q^{\lambda'}}}\big(kN^{K^{\lambda'}}_{G'}\big(Q^{\lambda'}\big)\big)_{\beta'}\big)$$

determined by the local point $\bar{\varepsilon}$ and the embedding $\tilde{h}^{\ddot{\beta},\beta'}_{\hat{\bar{\varepsilon}}}(Q)$ (cf. Proposition 2.8 and (4.13.1)), the $kN^K_P(Q)$-interior algebra

$$\mathrm{Ind}_{\rho^{K^\lambda}_{Q^\lambda}}\big(\ddot{D}_{\ddot{\beta}} \otimes_k \mathrm{Res}_{\rho^{K^{\lambda'}}_{Q^{\lambda'}}}\big(kN^{K^{\lambda'}}_{G'}\big(Q^{\lambda'}\big)\big)_{\beta'}\big)_{\hat{\bar{\varepsilon}}}$$

is a 0-split $\mathfrak{D}(N^K_P(Q) \times N^K_P(Q))$-module and we have a $kN^K_P(Q)$-interior algebra isomorphism

$$\big(kN^K_G(Q)\big)_{\bar{\varepsilon}} \cong \mathbb{H}_0\big(\mathrm{Ind}_{\rho^{K^\lambda}_{Q^\lambda}}\big(\ddot{D}_{\ddot{\beta}} \otimes_k \mathrm{Res}_{\rho^{K^{\lambda'}}_{Q^{\lambda'}}}\big(kN^{K^{\lambda'}}_{G'}\big(Q^{\lambda'}\big)\big)_{\beta'}\big)_{\hat{\bar{\varepsilon}}}\big).$$

Consequently, it follows from Theorem 18.8 in [6] that the block algebras $kN^K_G(Q)f$ and $kN^{K^{\lambda'}}_{G'}(Q^{\lambda'})f^{\lambda'}$ are *Rickard equivalent*.

4.17. Moreover, from our *basic* hypothesis, we know that $\ddot{A}_{\ddot{\varepsilon}}$ admits $\ddot{R} \times_R \ddot{R}$- and $\ddot{R} \times_{R'} \ddot{R}$-stable bases $\ddot{Y}$ and $\ddot{Y}'$ where $\mathrm{Ker}(\rho) \times \mathrm{Ker}(\rho)$ and $\mathrm{Ker}(\rho') \times \mathrm{Ker}(\rho')$ respectively act freely; then, it is elementary to check that $\mathrm{Br}_{Q^\lambda}(\ddot{Y}^{Q^\lambda})$ and $\mathrm{Br}_{Q^\lambda}(\ddot{Y}'^{Q^\lambda})$ are bases of $\ddot{D} = \ddot{A}_{\ddot{\varepsilon}}(Q^\lambda)$, and, since we have

$$N^{K^\lambda}_{\ddot{R}}\big(Q^\lambda\big) = C_{\ddot{R}}\big(Q^\lambda\big) \cdot \Delta_\lambda\big(N^K_R(Q)\big),$$

it is not difficult to check that $N^{K^\lambda}_{\ddot{R}}(Q^\lambda) \times_{N^K_R(Q)} N^{K^\lambda}_{\ddot{R}}(Q^\lambda)$ stabilizes and $N^{K^\lambda}_{\mathrm{Ker}(\rho)}(Q^\lambda) \times N^{K^\lambda}_{\mathrm{Ker}(\rho)}(Q^\lambda)$ acts freely on the basis $\mathrm{Br}_{Q^\lambda}(\ddot{Y}^{Q^\lambda})$; similarly, $N^{K^\lambda}_{\ddot{R}}(Q^\lambda) \times_{N^{K^{\lambda'}}_{R^{\lambda'}}(Q^{\lambda'})} N^{K^\lambda}_{\ddot{R}}(Q^\lambda)$ stabilizes and $N^{K^\lambda}_{\mathrm{Ker}(\rho')}(Q^\lambda) \times N^{K^\lambda}_{\mathrm{Ker}(\rho')}(Q^\lambda)$ acts freely on the basis $\mathrm{Br}_{Q^\lambda}(\ddot{Y}'^{Q^\lambda})$. In conclusion, the block algebras $kN^K_G(Q)f$ and $kN^{K^{\lambda'}}_{G'}(Q^{\lambda'})f^{\lambda'}$ are *basic Rickard equivalent*.

Lemma 4.18. *Let (Q, f) be a Brauer (b, G)-pair, K a subgroup of the image of $N_G(Q, f)$ in* $\mathrm{Aut}(Q)$ *and ν the point of $Q \cdot N^K_G(Q)$ over kGb such that $\mathrm{Br}_Q(\nu) = \{f\}$. For any local pointed group R_ε over kGb such that $Q \subset R$ and $R_\varepsilon \subset Q \cdot N^K_G(Q)_\nu$, there is a local point $\bar{\varepsilon}$ of $N^K_R(Q)$ over $kN^K_G(Q)f$ such that the structural homomorphism from $N^K_G(Q)$ to $N^K_{kG}(Q)$ induces a $kN^K_R(Q)$-interior algebra embedding*

$$\big(kN^K_G(Q)\big)_{\bar{\varepsilon}} \to \bar{N}^K_{(kG)_\varepsilon}(Q). \tag{4.18.1}$$

Moreover, $N_R^K(Q)_{\bar\varepsilon}$ is a maximal local pointed group over $kN_G^K(Q)f$ if and only if R_ε is a defect pointed group of $Q\cdot N_G^K(Q)_\nu$. In particular, $N_R^K(Q)$ is a defect group of f as a block of $N_G^K(Q)$ if and only if R is a defect group of f as a block of $Q\cdot N_G^K(Q)$.

Proof. As usual, we identify $(kG)(Q)$ with $kC_G(Q)$; from the existence of stable bases, it is easily checked that $\mathrm{Br}((kG)^H)=kC_G(Q)^H$ for any subgroup H of $N_G(Q)$ containing Q; hence, for any point β of H over kG, if $\mathrm{Br}_Q(\beta)\neq\{0\}$ then $\mathrm{Br}_Q(\beta)$ is a point of H over $kC_G(Q)$ and, conversely, any point of H over $kC_G(Q)$ can be lifted to $(kG)^H$; note that, since $Q\cdot N_G^K(Q)$ contains $C_G(Q)$, $kC_G(Q)^{Q\cdot N_G^K(Q)}$ is contained in $Z(kC_G(Q))$, which guarantees the existence of ν. Thus, since $\mathrm{Br}_R(\varepsilon)\neq\{0\}$ and $R=Q\cdot N_R^K(Q)$, $\mathrm{Br}_Q(\varepsilon)$ is a local point of both R and $N_R^K(Q)$ over $kC_G(Q)$, and the inclusion of R_ε in $Q\cdot N_G^K(Q)_\nu$ forces

$$f\,\mathrm{Br}_Q(\varepsilon)=\mathrm{Br}_Q(\varepsilon).$$

In particular, choosing $i\in\varepsilon$ such that $(kG)_\varepsilon=i(kG)i$, $\mathrm{Br}_Q(i)$ is a primitive idempotent in $(kC_G(Q)f)^{N_R^K(Q)}$ and, since the inclusion of $C_G(Q)$ in $N_G^K(Q)$ induces an injective homomorphism

$$\big(kC_G(Q)f\big)\big(N_R^K(Q)\big)\to\big(kN_G^K(Q)f\big)\big(N_R^K(Q)\big),\tag{4.18.2}$$

there is a primitive idempotent j in $(kN_G^K(Q)f)^{N_R^K(Q)}$ fulfilling

$$j\,\mathrm{Br}_Q(i)=j=\mathrm{Br}_Q(i)j\quad\text{and}\quad\mathrm{Br}_{N_R^K(Q)}^{kN_G^K(Q)f}(j)\neq 0.$$

That is to say, j belongs to a local point $\bar\varepsilon$ of $N_R^K(Q)$; moreover, it is clear that the $kN_R^K(Q)$-interior algebra isomorphism $kN_G^K(Q)\cong\bar N_{kG}^K(Q)$ [7, Proposition 3.5] maps $(kN_G^K(Q))_{\bar\varepsilon}$ bijectively onto $j\bar N_{kG}^K(Q)j$, whereas we have

$$N_{(kG)_\varepsilon}^K(Q)=\bigoplus_{\varphi\in K}\big(i(kG)i\big)\big(\Delta_\varphi(Q)\big)=\mathrm{Br}_Q(i)\bar N_{kG}^K(Q)\,\mathrm{Br}_Q(i)$$

and embedding (4.18.1) easily follows.

Conversely, for any local point $\bar\varepsilon'$ of $N_R^K(Q)$ over $kN_G^K(Q)f$, it is clear that there are $j'\in\bar\varepsilon'$ and a primitive idempotent $\bar\imath'$ in

$$\big(kC_G(Q)f\big)^{N_R^K(Q)}=\big(kC_G(Q)f\big)^R$$

fulfilling $j'\bar\imath'=j'=\bar\imath'j'$, and therefore we have $\bar\imath'=\mathrm{Br}_Q(i')$ for a suitable primitive idempotent in $(kGb)^R$ belonging to a local point ε' of R over kGb such that $R_{\varepsilon'}\subset Q\cdot N_G^K(Q)_\nu$. Hence, since we have $R=Q\cdot N_R^K(Q)$, so that R and $N_R^K(Q)$ determine each other, and all the maximal local pointed groups contained in a pointed group are mutually conjugate, it is easily checked that $N_R^K(Q)_{\bar\varepsilon}$ is a maximal local pointed group over $kN_G^K(Q_\delta)f$ if and only if R_ε is a maximal local pointed group over $Q\cdot N_G^K(Q_\delta)_\nu$. The last statement follows from this one applied once again to $K\cdot\mathrm{Inn}(Q)$. □

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